Harley Flanders

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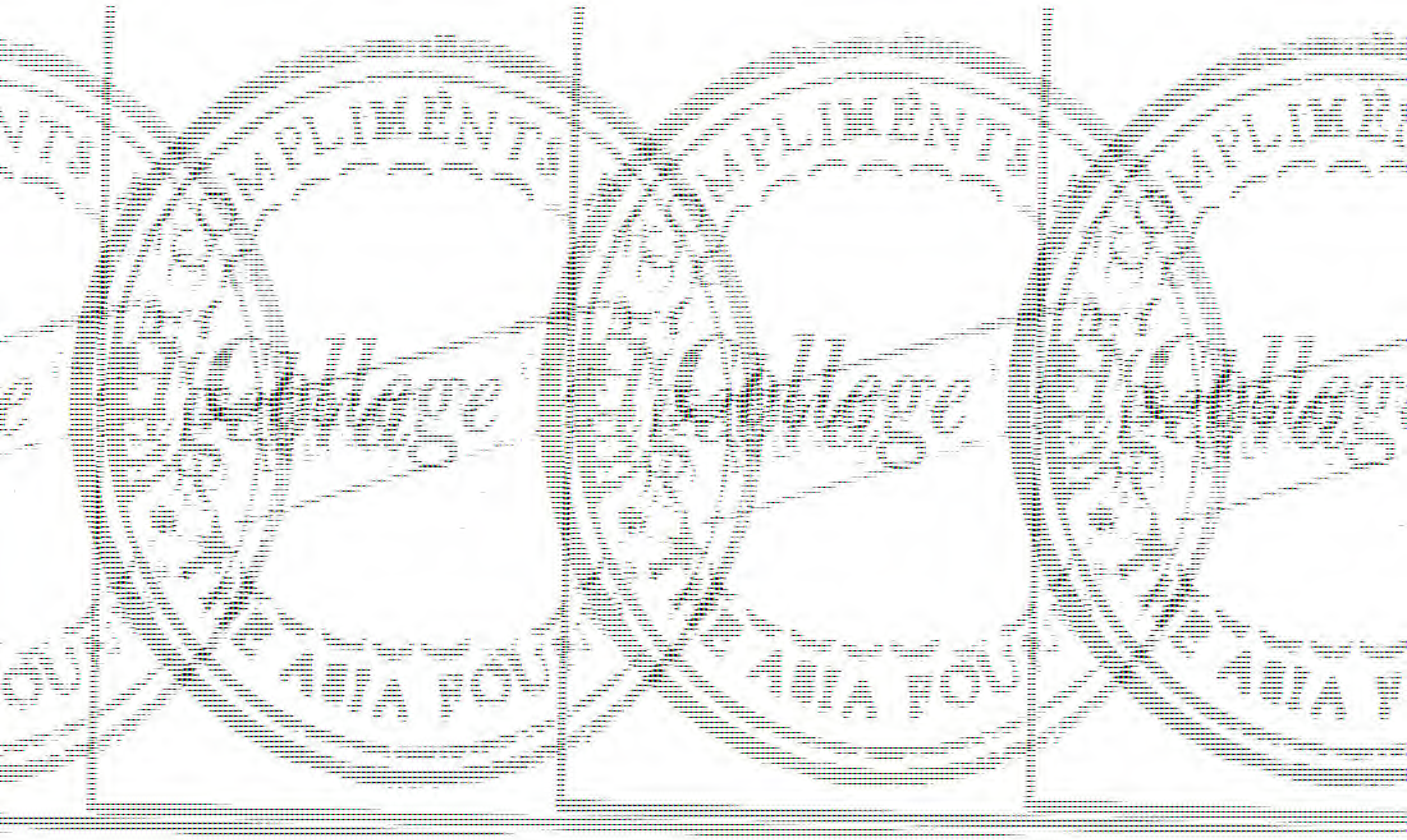
COMPLIMENTS

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Algebra

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Preface

The principal objectives of the author in writing this college algebra textbook are clarity of presentation and carefully graded sets of diversified problems covering a wide range of difficulty. The book should prove especially beneficial to students who have an inadequate mathematical background and to those who have not yet acquired the habit of orderly and independent thinking. For the better-than-average student there are numerous stated problems many of which will test the mettle of the very best pupils. Some features of the text are:

1. A thorough review of high school algebra is provided for students who have had inadequate high school instruction and for those who have had several years in which to forget their previous training.

2. Student difficulties are anticipated by pointing out and discussing common errors. The duty of the instructor is, not only to teach correct methods, but also to convince the student of the error in each incorrect operation.

3. There are more than 2800 problems, including an ample number of medium difficulty. More than 300 stated problems provide an effective challenge to the better student.

4. Answers to three-fourths of the problems are given in the back of the book. No answers are given, either in the text or in separate pamphlet form, for problems numbered 4, 8, 12, etc. The author believes this plan is preferable to the common practice of listing in

the text the answers to half of the problems and printing in a separate booklet the answers to the other half. The problems in each exercise are so arranged that good coverage may be obtained by assigning numbers 1, 5, 9, etc. or similar groups starting with 2, 3, or 4.

5. The definition of a function has been simplified in an attempt to prevent the student from getting the idea that an equation, such as $2x + 3y = 6$, is a function of x.

6. In solving a system of equations by graphic methods, some algebra books write the solution in terms of fractions, e.g. $x = \frac{1}{7}$, $y = \frac{3}{11}$. Such a result was obviously not obtained graphically. The author has tried to be "honest" in his treatment of graphic solutions. Non-integral solutions have been approximated by the use of decimals. A definite statement is made regarding the amount of tolerance.

7. In graphing, general instructions are given as to how to choose the values to be assigned to the independent variable.

8. The new characteristic rule for logarithms has been proved by classroom experiment to be effective, especially in finding a number from its logarithm. Instructors who prefer the old rule will find it listed as an alternative.

The author wishes to thank the McGraw-Hill Book Company for permission to use passages from his *Plane Trigonometry*, copyright, 1942, and to acknowledge his indebtedness to his sister, Mrs. E. N. Hetzel, for her helpful advice. He wishes to thank his colleagues, Professors J. N. Michie, H. E. Woodward, R. S. Underwood, E. A. Hazlewood, and F. W. Sparks for suggestions and assistance.

E. Richard Heineman

Lubbock, Texas

Contents

A note to the student

A mastery of algebra requires (1) a certain amount of memory work and (2) a great deal of practice and drill in order to acquire experience and skill in the application of the memory work. Your instructor is a "trouble-shooter" who attempts to prevent you from going astray, supplies missing links in your mathematical background, and tries to indicate the "common sense" approach to the problem. The memory work in any course is one thing that the student can and should perform by himself. The least you can do for your instructor and yourself is to *commit to memory each definition and theorem as soon as you contact it.* This can be accomplished most rapidly, not by reading, but by writing the definition or theorem until you can reproduce it without the aid of the text.

In working the problems, do not continually refer back to the illustrative examples. Study the examples so thoroughly (by writing them) that you can reproduce them with your text closed. Only after the examples are entirely clear and have been completely mastered should you attempt the unsolved problems. These problems should be worked *without referring to the text.*

College Algebra

chapter 1 Introductory topics

1. The fundamental algebraic operations. The four fundamental operations of algebra are addition, subtraction, multiplication, and division.

The **sum** of two or more numbers is the result of adding them. The plus sign, $+$, is used to indicate addition.

Illustration 1. The sum of 2, 7, and 8 is indicated by $2+7+8=17$.

Illustration 2. The sum of the quantities a and b is indicated by $a + b$.

The **difference** of two numbers, a and b, is the result of subtracting the second number from the first. The minus sign, $-$, is used to indicate subtraction.

Illustration 3. The difference of 8 and 2 is $8 - 2 = 6$.

The **product** of two or more numbers is the result of multiplying them. Each of the given numbers is called a **factor** of their product. Multiplication is indicated by the use of a cross, $\times$, or a dot, or by mere juxtaposition.

Illustration 4. The product of 2, 3, and 5 is $2 \cdot 3 \cdot 5 = 30$. The factors of 30 are 2, 3, and 5.

Illustration 5. The product of 7, a, and b is indicated by $7ab$.

The **quotient** of a by b, where b is not 0, is the result of dividing a by b. This division is indicated by writing $a \div b$, or $\frac{a}{b}$, or a/b. If

the quotient of a by b is c, then b times c must equal a. In other words, if $\frac{a}{b} = c$, then $bc = a$. The **dividend** is the number a into which we are dividing the number b, which is called the **divisor.** In the fraction $\frac{a}{b}$, a is called the **numerator** and b is called the **denominator.** The quotient $a \div b$, or the fraction a/b, is frequently called the **ratio** of a to b.

Illustration 6. The quotient of 20 by 4 is 5; or, the ratio of 20 to 4 is 5.

2. The number system. In the process of counting, we employ such numbers as 1, 2, 3, 4, etc. They are called **positive integers.** If two of these numbers are *added* or *multiplied*, the result is another positive integer.

In order to make it possible to *subtract* a positive integer from a smaller one, we extend the number system to include **negative integers.** Thus, $5 - 8 = -3$. Moreover, **zero** is included as a number to permit the subtraction of a number from itself.

The division of any integer by any other integer is made possible by again extending the number system to include *rational numbers* (ordinary fractions), such as $\frac{2}{3}$, $-\frac{5}{7}$, $\frac{9}{4}$. In general,

A **rational number** *is a number that can be expressed as the quotient of two integers.* It is obvious that all integers are rational numbers. Why? It can be shown that all ending decimals and all nonending, repeating decimals are rational. For example, the ending decimal 1.125 is rational because it is equal to $\frac{9}{8}$. And the nonending, repeating decimal $.2727\cdots$* is rational because it can be expressed as the quotient of 3 by 11.

A positive or negative number that cannot be expressed as the quotient of two integers is called an **irrational number.** For example, $\sqrt{2}$, $\sqrt[3]{4}$, π, $1 + \sqrt{7}$ are irrational.

All of the afore-mentioned numbers belong to the class of **real numbers.** Included are all positive numbers (rational and irrational), all negative numbers, and zero.

* The dots indicate that the couplet 27 is to be repeated indefinitely.

Another class of numbers, **imaginary numbers** (to distinguish them from real numbers), will be discussed in Art. 59.

A positive integer is said to be **prime** if it is not divisible by any positive integer except itself and 1. For example, the numbers 1, 2, 3, 5, 7, 11, 13, 17, etc., are prime. All positive integers that are not prime are called **composite.**

3. Laws governing algebraic operations. In forming an algebra that generalizes arithmetic, we assume the following laws:

I. The commutative law holds for addition and multiplication:

$$a + b = b + a.$$
$$ab = ba.$$

II. The associative law holds for addition and multiplication:

$$(a + b) + c = a + (b + c).$$
$$(ab)c = a(bc).$$

III. Multiplication is distributive with respect to addition:

$$a(b + c) = ab + ac.$$

4. Operations with the number zero. If a is any number, then

$$a + 0 = a, \qquad a - 0 = a, \qquad a(0) = 0.$$

If $a \neq 0$,* then $\dfrac{0}{a} = 0$ because $a \cdot 0 = 0$.

Division by 0 is ruled out. When we write $\frac{6}{2}$, we ask, "How many 2's add up to 6?" When we write $\frac{6}{0}$, we ask, "How many 0's will add up to 6?" Such a question is obviously absurd.

Let us recall that division can always be checked by showing that the dividend is equal to the product of the divisor and the quotient. If $\frac{6}{0}$ equals a number b, then $(0)(b)$ must equal 6. No such number b exists. We conclude that division by 0 must be ruled out.

5. Graphic representation of the real numbers. Let point A in Fig. 1 represent the number 0. All positive numbers are represented in order of size by the points to the right of A. All negative numbers

* The symbol $\neq$ is read "is not equal to."

are represented in order by points to the left of A. If a point B lies to the right of a point C, then we shall say that the number represented by B is *algebraically greater* than the number represented

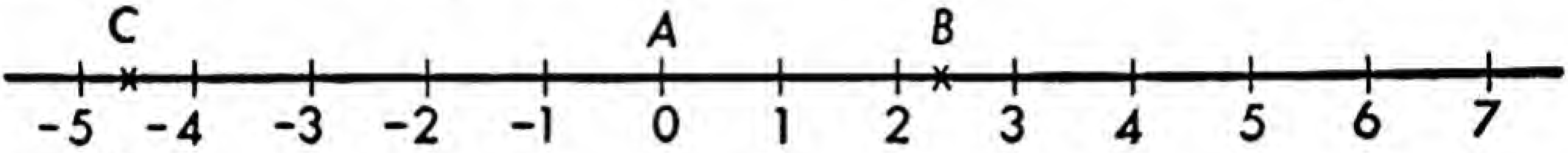


FIG. 1

by C. We indicate this by writing $B > C$ or $C < B$. The statement $B > C$ is read, "B is greater than C." For example, $3 > -5$, $-2 > -3$, $0 > -1$, $-3 < 1$.

When no sign is placed in front of a number, the positive sign is tacitly assumed. Thus, 7 means $+7$.

The negative of a number is obtained by changing the sign of the number. The negative of 4 is -4; the negative of -4 is 4.

The **absolute** (or **numerical**) **value** of a positive number or of 0 is the number itself. The absolute value of a negative number is the number with its sign changed. The absolute value of a number a is usually designated by $|a|$.

Illustration 1. The absolute value of 3 is 3. The absolute value of -3 is 3; or, $|-3| = 3$.

Illustration 2. $5 > -7$ but $|5| < |-7|$. This means that 5 is algebraically greater than -7, but 5 is numerically less than -7.

6. Laws of signs. To add two numbers with like signs, add their absolute values and prefix their common sign.

Illustration 1. $(-6) + (-4) = -10$; $(+2) + (+3) = +5$.

To add two numbers with unlike signs, subtract the smaller absolute value from the larger and prefix the sign of the number having the larger absolute value.

Illustration 2. $(-1) + 9 = 8$; $5 + (-7) = -2$.

To subtract a number, add its negative.

Illustration 3. $14 - (-3) = 14 + 3 = 17$; $(-6) - (-5) = -6 + 5 = -1$.

To multiply (or divide) two numbers with like signs, multiply (or divide) their absolute values and prefix a plus sign.

To multiply (or divide) two numbers with unlike signs, multiply (or divide) their absolute values and prefix a minus sign.

Illustration 4.

$$(-4)(-6) = 24;\ (-2)(3) = -6;\ \frac{-15}{-5} = 3;\ \frac{14}{-2} = -7.$$

Exercise 1

Identify the following real numbers as rational or irrational.

1. $\frac{4}{5}$, -7, $\sqrt{5}$, $.333\cdots$, 3.2, $\sqrt{9}$, 0.
2. 1776, $-\frac{5}{8}$, $.029$, $\sqrt{3}$, $1.666\cdots$, $\sqrt{144}$, $\frac{4}{25}$.

Insert the proper sign, < or >, between each of the following pairs of numbers.

3. -6 and -1. 4. 1 and -7. 5. 0 and -3. 6. -2 and -3.

Perform the indicated operations.

7. $(-7) + (-3) - (6)$. 8. $17 - (-2) + 1$.

9. $3(-4)(-5)$. 10. $(-2)(-3)(-4)$.

11. $\dfrac{18}{-6}$. 12. $\dfrac{-24}{4}$. 13. $\dfrac{-30}{-10}$. 14. $\dfrac{12}{-2}$.

Find the sum, the difference, the product, and the quotient of each of the following pairs of numbers.

15. -6 and 1. 16. 3 and -3.

17. -8 and -2. 18. -12 and 4.

19. 0 and -9. 20. -6 and -2.

21. 10 and -5. 22. -7 and 0.

23. State the value of (*a*) $\frac{0}{8}$, (*b*) $\frac{8}{0}$, (*c*) $(0)(8)$.

24. Show that the repeating decimal $.142857142857\cdots$ is rational because it can be expressed as the quotient of 1 by 7.

25. If $\dfrac{a}{b} = 0$, what can be said about a or b?

26. Find the fallacy in the following "proof that $2 = 1$."

Let	$a = 1$
Multiply both sides by a:	$a^2 = a$
Subtract 1 from both sides:	$a^2 - 1 = a - 1$
Factor:	$(a + 1)(a - 1) = a - 1$
Divide both sides by $(a - 1)$:	$a + 1 = 1$
Since $a = 1$,	$2 = 1$.

7. Terms, factors, coefficients. A **term,** or **monomial,** consists of quantities combined by only multiplications and divisions. If two or more terms are connected by plus or minus signs, the expression formed is called a **polynomial.** A **binomial** is a polynomial consisting of two terms. A **trinomial** is a polynomial containing three terms.

Illustration 1. $7abc$ is a term. $3x + 7y$ is not a term; it consists of two terms; it is a binomial.

A **factor** of an expression is a quantity which, when multiplied by one or more other quantities, will produce the expression. A considerable part of the difficulties encountered by a beginner in algebra is due to his failure to distinguish between term and factor.

Illustration 2. a is a factor of ab; but a is not a factor of $(a + b)$; a is a factor of $a(x + y)$. What is the other factor?

Illustration 3. The expression $7ab$ has the following factors: 7, a, b, $7a$, $7b$, and ab, in addition to the obvious factors 1 and $7ab$. The prime factors of $30x$ are 2, 3, 5, and x.

Illustration 4. The expression $(2x + 3y)(4r + 5s + 6t)$ consists of two factors. The first factor, $2x + 3y$, contains two terms. The second factor, $4r + 5s + 6t$, consists of three terms.

In a given term, any factor is called the **coefficient** of the product of the other factors.

Illustration 5. In the term $7ab$, we say that 7 is the coefficient of ab; a is the coefficient of $7b$; b is the coefficient of $7a$; $7a$ is the coefficient of b; etc.

In a term that is the product of a number and one or more letters, we refer to the number as the **numerical coefficient,** or simply *the* coefficient of the term.

Illustration 6. In the term $7ab$, the numerical coefficient or the coefficient is 7. In the term $-8a$, the numerical coefficient is -8. In the term x, the coefficient is 1.

If two terms are the same except for the numerical coefficient, they are called **like terms.** For example, $5ab$ and $-4ab$ are like terms. Like terms can be added to form a single term. The sum of unlike terms can only be indicated.

Illustration 7. $x + 2x + 3x = 6x$. $a + 2b$ cannot be combined.

8. Symbols of grouping. If we wish to indicate that several terms are to be treated as a single expression, we use parentheses ( ), or brackets [ ], or braces { }, or a vinculum —. For example, $a-(b+c)$ means that the quantity $(b+c)$ is to be subtracted from a. This is equivalent to first subtracting b from a, then subtracting c from this result: $a-b-c$. When removing parentheses preceded by a *minus* sign, rewrite each term involved *with its sign changed.* When removing parentheses preceded by a plus sign, merely rewrite each term involved without changing its sign.

Illustration 1. $2a-(b-c+d)=2a-b+c-d$.

Illustration 2. $5a+(b-c)=5a+b-c$.

When one pair of symbols encloses another pair, it is customary to remove the innermost pair first.

Illustration 3.

$$\begin{aligned} 9a-[7b-5c-(b-c)] &= 9a-[7b-5c-b+c] \\ &= 9a-[6b-4c]=9a-6b+4c. \end{aligned}$$

9. Exponents. We use the symbol a^3 as a short way of representing the product of three equal factors, each of which is a; that is, a^3 means $a\cdot a\cdot a$. In general, we define a^m to mean $a\cdot a\cdot a\cdots a$ (m factors). The symbol a^m is read, "the mth power of a," or, in short, "a to the mth." We call a the **base** and m the **exponent** of the power.

The following laws of exponents will be proved and discussed in Art. 46. They should be memorized.

[1] $\quad a^m\cdot a^n=a^{m+n}.$ $\qquad$ *Example:* $2^4\cdot 2^3=2^7$.

[2] $\quad (a^m)^n=a^{mn}.$ $\qquad$ *Example:* $(2^4)^3=2^{12}$.

[3] $\quad \dfrac{a^m}{a^n}=a^{m-n}.$ $\qquad$ *Example:* $\dfrac{2^8}{2^2}=2^6$.

[4] $\quad (ab)^n=a^nb^n.$ $\qquad$ *Example:* $(2b)^3=8b^3$.

[5] $\quad \left(\dfrac{a}{b}\right)^n=\dfrac{a^n}{b^n}.$ $\qquad$ *Example:* $\left(\dfrac{a}{4}\right)^3=\dfrac{a^3}{64}$.

Exercise 2

Remove all symbols of grouping and simplify.

1. $9y+(5x-2)-(3y-8)-(x+y)$.
2. $6a-(a+3)+(2a-5)-(-a-1)$.

3. $5a - [3a + (b + 7) - (a + 5)]$.

4. $8x - [7y - (6x - 5) - \overline{y - 2}]$.

5. $x - \{2y - [3x - (4y - 5)]\}$.

6. $a + \{2b - [3a - (4b + 5)]\}$.

Compute by using the definition of an exponent.

7. 5^3. **8.** 7^2. **9.** $(-2)^5$. **10.** $(-10)^6$.

11. $(-.1)^4$. **12.** $(-.1)^5$. **13.** $(\frac{3}{7})^2$. **14.** $(\frac{2}{5})^3$.

Perform the indicated operations by using the laws of exponents.

15. $a^4 \cdot a^5$. **16.** $x^2 \cdot x^7$. **17.** $\dfrac{x^9}{x^3}$. **18.** $\dfrac{a^8}{a^2}$.

19. $(a^4)^3$. **20.** $(x^7)^2$. **21.** $(2xy)^4$. **22.** $(3ab)^3$.

23. $(-10x^3)^2$. **24.** $(-x^7)^4$. **25.** $\left(-\dfrac{a}{b}\right)^7$. **26.** $\left(-\dfrac{2a}{b^2}\right)^5$.

27. $(2ab^2c^5)^3$. **28.** $(6xy^3z^4)^2$. **29.** $\dfrac{a(a^5)^6}{(a^4)^7}$. **30.** $\dfrac{(a^9)^2}{a^3(a^2)^4}$.

For each of the following expressions, (a) state the number of terms, (b) list the factors of each term, (c) state the coefficient of each term.

31. xy^2. **32.** $3r + 8$. **33.** $5x + y + 12$. **34.** $t(x + y)$.

35. In the term $24x^2y^3$, state the coefficient of (*a*) x^2y^3, (*b*) x^2, (*c*) $8xy^3$.

36. In the term $30xy^2$, state the coefficient of (*a*) xy^2, (*b*) $5y^2$, (*c*) $30y$.

10. Multiplication. Two monomials are multiplied by using the commutative and associative laws and the law of exponents, $a^m \cdot a^n = a^{m+n}$.

Illustration 1. $(7a^3b^2)(5a^4b^6) = (7 \cdot 5)(a^3 \cdot a^4)(b^2 \cdot b^6) = 35a^7b^8$.

To find the product of a monomial by a polynomial, use the distributive law together with the previously mentioned laws.

Illustration 2.

$$\begin{aligned} -3a^2b(5a^7 - 4ab^4) &= (-3a^2b)(5a^7) + (-3a^2b)(-4ab^4) \\ &= -15a^9b + 12a^3b^5. \end{aligned}$$

To multiply one polynomial by another, multiply each term of one polynomial by all the terms of the other and add the partial

products. It is usually desirable to arrange the polynomials in ascending or descending powers of some letter involved.

Example 1. Multiply $(3x^2 - 4x - 5)$ by $(2x - 7)$.

Solution.

$$\begin{array}{lrl}
 & 3x^2 - \ \ 4x - \ \ 5 & \\
 & 2x - \ \ 7 \qquad\quad & \\
\hline
 & 6x^3 - \ \ 8x^2 - 10x \qquad\quad & \text{(Multiplying by } 2x) \\
 & - 21x^2 + 28x + 35 & \text{(Multiplying by } -7) \\
\hline
\textit{Product:} & 6x^3 - 29x^2 + 18x + 35 & \text{(Adding the partial products)}
\end{array}$$

11. Division. To divide one monomial by another, use the law of exponents, $\dfrac{a^m}{a^n} = a^{m-n}$.

Illustration 1. $\dfrac{-12x^7y^9}{3x^2y^3} = \dfrac{-12}{3} \cdot \dfrac{x^7}{x^2} \cdot \dfrac{y^9}{y^3} = -4x^5y^6.$

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Illustration 2.

$$\frac{6x^5 + 9x^4 - 15x^2}{3x^2} = \frac{6x^5}{3x^2} + \frac{9x^4}{3x^2} - \frac{15x^2}{3x^2} = 2x^3 + 3x^2 - 5.$$

To divide one polynomial by another: (1) arrange each of them in descending (or ascending) powers of some common letter, (2) divide the first term of the dividend by the first term of the divisor, thus obtaining the first term of the quotient, (3) multiply the entire divisor by the first term of the quotient and subtract this product from the dividend, (4) consider this remainder as a new dividend and repeat steps 2 and 3, etc.

At any stage of the division, the partial quotient at that point and the corresponding remainder always satisfy the equation

$$\frac{\textbf{dividend}}{\textbf{divisor}} = \textbf{quotient} + \frac{\textbf{remainder}}{\textbf{divisor}} \tag{1}$$

or

$$\textbf{dividend} = (\textbf{divisor})(\textbf{quotient}) + \textbf{remainder.} \tag{2}$$

For example, $\frac{25}{8} = 3 + \frac{1}{8}$ or $25 = 8 \cdot 3 + 1$.

Example 1. Divide $(12a^4 + 11a^3 + 18a - 10)$ by $(4a^2 + 5a - 1)$.

Solution.

$$
\begin{array}{rl}
 & 3a^2 - \quad a + 2 = \textit{quotient} \\
\textit{divisor} = \underline{4a^2 + 5a - 1} & \overline{\big)\, 12a^4 + 11a^3 \qquad\qquad + 18a - 10} = \textit{dividend} \\
\text{(subtract)} & \underline{12a^4 + 15a^3 - 3a^2} \\
 & \qquad - 4a^3 + 3a^2 + 18a \\
 & \qquad \underline{- 4a^3 - 5a^2 + \quad a} \\
 & \qquad\qquad\qquad 8a^2 + 17a - 10 \\
 & \qquad\qquad\qquad \underline{8a^2 + 10a - \enspace 2} \\
 & \qquad\qquad \textit{remainder} = 7a - \enspace 8
\end{array}
$$

Writing our results in the form of equation (1), we have

$$\frac{12a^4 + 11a^3 + 18a - 10}{4a^2 + 5a - 1} = 3a^2 - a + 2 + \frac{7a - 8}{4a^2 + 5a - 1}.$$

The division can be checked by multiplying the quotient by the divisor and then adding the remainder. This result should equal the dividend. As a partial check, we can set a equal to some number, such as 2, and show that equation (2) holds. For $x = 2$, we have: *dividend* $= 306$, *divisor* $= 25$, *quotient* $= 12$, *remainder* $= 6$ Our partial check gives us

$$306 = 25 \cdot 12 + 6.$$

Since the numbers do check, the work is probably correct. In case the numbers do not check, we can be certain that there is an error in the work or in the check.

Exercise 3

Perform the indicated multiplications.

1. $(-2a^5)(-3a^4)(5a)$.
2. $(2ab)b$.
3. $(-7a)(-2a^3b^2)(-ab^5)$.
4. $(3a^2b^2)b(-4a^4b)$.
5. $6a(2a - b)$.
6. $-3x(2x - 4y)$.
7. $-2a^2(3a - 4b - 5ab^3)$.
8. $5a(a^2 + 3a - 5)$.
9. $(x + 2)(3x^2 + 4x + 5)$.
10. $(3x + 4)(5x + 6)$.
11. $(4x - 3)(x^2 - 8x - 6)$.
12. $(2x - 1)(x^2 + 4x + 2)$.
13. $(5x - 2)(3x^2 + 7x - 2)$.
14. $(x + 1)(x^3 - x^2 + x - 1)$.
15. $(x^2 + 3x + 1)(x^2 - 3x + 1)$.
16. $(x^2 + x + 1)(x^3 - x + 1)$.

Perform the indicated divisions. Check as directed by the instructor.

17. $\dfrac{10a^4b^6c^8}{2ab^6c^2}$.

18. $\dfrac{-12x^{10}y^9z^8}{4x^2y^3z}$.

19. $\dfrac{10x^3 - 15x^4 - 20x^5}{-5x^3}$.

20. $\dfrac{4a^4 + 6a^3 - 10a^2}{2a^2}$.

21. $\dfrac{10x^2 - 7x - 12}{5x + 4}$.

22. $\dfrac{13x + 12x^2 - 14}{3x - 2}$.

23. $\dfrac{5x^2 - 6x - 7 + x^3}{x - 2}$.

24. $\dfrac{2x^3 - x^2 - x + 8}{x + 3}$.

25. $\dfrac{6a^4 - 13a^3 - 9a^2 + 29}{2a - 5}$.

26. $\dfrac{12x^4 - 14x^3 - 19x + 1}{3x + 1}$.

27. $\dfrac{7x^3 - 24x^2 + 17x + 5}{x^2 - 3x + 1}$.

28. $\dfrac{x^5 - 4x^3 - 11x + 19 + 16x^2}{2x - x^2 - 3}$.

29. $\dfrac{-15x^3 + 28 - 45x + 38x^2}{3x - 4}$.

30. $\dfrac{8x^3 + 27y^{15}}{2x + 3y^5}$.

31. $\dfrac{x^5 - 32y^5}{x - 2y}$.

32. $\dfrac{10x^4 - 11x^3y + 22x^2y^2 - 4xy^3 + 16y^4}{5x^2 + 2xy + 4y^2}$.

chapter 2

Special products and factoring

12. Special products. The product of any two polynomials can be found by the general method given in Art. 10. The product of two binomials occurs so frequently that we shall use the following time-saving devices to compute these special products.

[1] *The product of the sum and difference of two numbers is equal to the difference of their squares:*

$$(a + b)(a - b) = a^2 - b^2.$$

Illustration 1. $(4x + 7y)(4x - 7y) = 16x^2 - 49y^2$.

[2] *The square of a binomial is equal to the square of the first term, plus twice the product of the terms,* plus the square of the second term:*

$$(a + b)^2 = a^2 + 2ab + b^2,$$
$$(a - b)^2 = a^2 - 2ab + b^2.$$

Illustration 2. $(3x + 5y)^2 = 9x^2 + 30xy + 25y^2$.

$$\left(6x - \frac{1}{7}\right)^2 = 36x^2 - \frac{12x}{7} + \frac{1}{49}.$$

[3] Products of the form $(ax + b)(cx + d)$ are computed by the method illustrated in the following examples.

Example 1. Multiply: $(2x - 3y)(4x + 5y)$.

Solution. Instead of writing one factor under the other as in

* In the binomial $(a - b)$, the second term is $(-b)$.

Art. 10, we shall use the following device to find the product mentally:

$$\begin{array}{l} \qquad 8x^2 \qquad\qquad -15y^2 \\ (2x - 3y)(4x + 5y) = 8x^2 - 2xy - 15y^2 \\ \qquad\quad -12xy \\ \qquad\quad 10xy \end{array}$$

It is to be noted that the product is written immediately with only one intermediate step: the two cross products, $10xy$ and $-12xy$, are kept in mind and added mentally to produce the middle term, $-2xy$.

Example 2. Multiply: $(3x - 5)(4x - 7)$.

Solution.

$$(3x - 5)(4x - 7) = 12x^2 - 41x + 35.$$

Exercise 4

Perform the indicated multiplications.

1. $(x + 3)(x - 3)$.
2. $(x + 7)(x - 7)$.
3. $(5x + 1)(5x - 1)$.
4. $(8x + 1)(8x - 1)$.
5. $(2y + 7)(2y - 7)$.
6. $(4y + 3)(4y - 3)$.
7. $(9x - 4y)(9x + 4y)$.
8. $(5x - 6y)(5x + 6y)$.
9. $(x + 6)^2$.
10. $(x + 5)^2$.
11. $(3x + 2)^2$.
12. $(7x + 4)^2$.
13. $(2t - 7)^2$.
14. $(6r - 1)^2$.
15. $(5x - 9y)^2$.
16. $(3x - 10y)^2$.
17. $(x + 6)(x + 2)$.
18. $(x + 4)(x + 5)$.
19. $(x - 7)(x - 1)$.
20. $(x - 3)(x - 8)$.
21. $(x + 9)(x - 2)$.
22. $(x - 5)(x + 7)$.
23. $(x - 3)(x + 1)$.
24. $(x - 6)(x + 4)$.
25. $(x + 2)(3x + 4)$.
26. $(x + 3)(5x + 7)$.
27. $(3y + 1)(7y + 2)$.
28. $(4y + 1)(3y + 8)$.
29. $(5x - 2y)(9x - y)$.
30. $(8x - 3y)(5x - 2y)$.
31. $(7x - 6)(5x - 4)$.
32. $(6x - 5)(4x - 3)$.
33. $(xy + 3a)(2xy - 7a)$.
34. $(2ab + 5t)(ab - t)$.

35. $(4x - y)(2x + 5y)$.

36. $(3x - 4y)(9x + 2y)$.

37. $(3x + 11)(4x - 7)$.

38. $(11x + 6)(5x - 3)$.

39. $(5r + 6s)(6r - s)$.

40. $(7w - 2z)(4w + 7z)$.

41. $(3x + \frac{1}{4})(3x - \frac{1}{4})$.

42. $(x + .2)(x - .2)$.

43. $(x + .7)^2$.

44. $(7x^3 + 4)^2$.

45. $(8x^4 - 5y^3)^2$.

46. $(5x - \frac{1}{6})^2$.

47. $[3(r + 2s)]^2$.

48. $[2(3t - 4w)]^2$.

49. $[r + 3s + 5t][r + 3s - 5t]$.

Solution. $[(r + 3s) + 5t][(r + 3s) - 5t] = (r + 3s)^2 - (5t)^2$
$= r^2 + 6rs + 9s^2 - 25t^2$.

50. $[(u + 2v) + 3w][(u + 2v) - 3w]$.

51. $[(u + v) + w]^2$.

52. $[r + (s + t)]^2$.

53. $[(r + s) - 2][(r + s) - 3]$.

54. $[(c + d) - 6][(c + d) + 7]$.

55. $[(r + s) + (t + u)][(r + s) - (t + u)]$.

56. $[a + 4b + t][a + 4b - t]$.

57. $[7r + s + 5t][7r - s + 5t]$.

58. $[3u + 4v - 5w][3u + 4v - 6w]$.

59. Expand $(r + s + t)^2$ and state the result in words.

13. Factoring. To factor a polynomial means to separate it into two or more quantities whose product is equal to the original polynomial. The following type forms can be used in factoring many expressions.

[1] *Common monomial factor:*

$$ax + ay = a(x + y).$$

Illustration 1. $5w^2 - 10mw + 5w = 5w(w - 2m + 1)$.

[2] *Difference of two squares:*

$$a^2 - b^2 = (a + b)(a - b).$$

Illustration 2. $121x^2 - y^6 = (11x + y^3)(11x - y^3)$.

[3] *Trinomials that are perfect squares:*

$$a^2 + 2ab + b^2 = (a + b)^2.$$
$$a^2 - 2ab + b^2 = (a - b)^2.$$

Illustration 3. $16x^2 - 72xy + 81y^2 = (4x - 9y)^2$.

Example 1. In the expression $25x^2 + (\quad) + 49y^2$, insert the proper middle term to form a perfect square.

Solution. If the given expression is to take the form $a^2 + 2ab + b^2$, then $a^2 = 25x^2$; hence $a = 5x$. Likewise $b = 7y$. The middle term must be $2ab = 2(5x)(7y) = 70xy$. The trinomial then becomes $25x^2 + 70xy + 49y^2$ which is the square of $(5x + 7y)$.

[4] $\boldsymbol{x^2 + qx + r = (x + a)(x + b)}$, **where** $\boldsymbol{ab = r}$ **and** $\boldsymbol{a + b = q}$.

Example 2. Factor $x^2 + 15x + 36$.

Solution. We seek two numbers whose product is 36 and whose sum is 15. Each of the following pairs of positive numbers has a product of 36: 1 and 36; 2 and 18; 3 and 12; 4 and 9; 6 and 6. Their sums are 37, 20, 15, 13, and 12, respectively. Obviously the desired numbers are 3 and 12. Hence

$$x^2 + 15x + 36 = (x + 3)(x + 12).$$

Example 3. Factor $x^2 - 4x - 77$.

Solution. We seek two numbers having a product of -77 and a sum of -4. These numbers are -11 and 7. Hence

$$x^2 - 4x - 77 = (x - 11)(x + 7).$$

[5]
$$\boldsymbol{px^2 + qx + r = (ax + b)(cx + d),}$$
where $\boldsymbol{ac = p,\quad bd = r,}$ **and** $\boldsymbol{ad + bc = q}$.

Example 4. Factor $5x^2 + 8x - 4$.

Solution. The term $5x^2$ tells us that the factors will take the form $(5x + b)(x + d)$. The term -4 has the following pairs of factors: 1 and -4; 2 and -2; 4 and -1. These three pairs of numbers can be fitted in the foregoing trial form in six ways with the following results:

$$\begin{aligned}
(5x + 1)(x - 4) &= 5x^2 - \mathbf{19x} - 4\\
(5x + 2)(x - 2) &= 5x^2 - \mathbf{8x} - 4\\
(5x + 4)(x - 1) &= 5x^2 - \mathbf{x} - 4\\
(5x - 1)(x + 4) &= 5x^2 + \mathbf{19x} - 4\\
(5x - 2)(x + 2) &= 5x^2 + \mathbf{8x} - 4\\
(5x - 4)(x + 1) &= 5x^2 + \mathbf{x} - 4.
\end{aligned}$$

Obviously our fifth trial was successful. Hence

$$5x^2 + 8x - 4 = (5x - 2)(x + 2).$$

With a little experience, one can factor an expression without first listing all the possibilities. In this particular problem, after rejecting the first possibility, we can see from the second product,

$$(5x + 2)(x - 2) = 5x^2 - \mathbf{8x} - 4$$

that only the sign of the middle term is incorrect. By merely changing the signs of the constant terms of the factors, we get

$$(5x - 2)(x + 2) = 5x^2 + \mathbf{8x} - 4.$$

Not all trinomials of the form $px^2 + qx + r$ are factorable if the numbers a, b, c, d are to be integers. For example, $x^2 + x + 4$ is not factorable.

Exercise 5

Factor by using forms [1] and [2].

1. $mr + ms$.
2. $bs - bt$.
3. $12x^2 - 3x$.
4. $20r^2 + 4r$.
5. $-4xy - 8xy^2$.
6. $-7ab + 21a^2b$.
7. $x^8 + x^6 + x^4$.
8. $x^7 + x^6 + x^5$.
9. $8a^3b^4 + 6a^2b^5 - 10a^8b^9$.
10. $9x^2y^7 - 12x^5y^6 + 6x^3y^4$.
11. $x^2 - 64$.
12. $x^2 - 100$.
13. $25x^2 - 36y^2$.
14. $81x^2 - 16y^2$.
15. $y^2 - \frac{1}{9}$.
16. $4y^2 - \frac{1}{49}$.
17. $28y^2 - 7z^2$.
18. $3c^2 - 75d^2$.
19. $16x^2 - y^{16}$.
20. $169x^2 - y^6$.

Factor by using form [3].

21. $x^2 + 8x + 16$.
22. $x^2 - 16x + 64$.
23. $x^2 - 20xy + 100y^2$.
24. $x^2 + 18x + 81$.
25. $36x^2 + 60xy + 25y^2$.
26. $49x^2 - 56xy + 16y^2$.
27. $x^{10} - 22x^5 + 121$.
28. $x^8 + 24x^4 + 144$.

Insert the proper middle term to form a perfect square; then factor.

29. $t^2 + (\quad) + 49$.
30. $w^2 - (\quad) + 36$.
31. $9x^2 - (\quad) + 4y^2$.
32. $25c^2d^2 + (\quad) + 121t^4$.

Factor by using forms [4] and [5].

33. $x^2 + 5x + 6$. 34. $x^2 + 17xy + 16y^2$. 35. $x^2 + 5x - 6$.
36. $x^2 + 3x - 70$. 37. $x^2 - 3x - 40$. 38. $t^2 - 2t - 48$.
39. $x^2 - 13xy + 36y^2$. 40. $x^2 - 15x + 44$. 41. $2x^2 + 3x + 1$.
42. $5x^2 + 14x + 9$. 43. $3r^2 + 5r - 8$. 44. $7x^2 + 5xy - 12y^2$.
45. $5x^2 - xy - 6y^2$. 46. $2x^2 - x - 1$. 47. $7x^2 - 18x + 11$.
48. $3a^2 - 7a + 4$. 49. $4x^2 - 24x + 35$. 50. $6x^2 + 31x + 35$.
51. $10x^2 + 11x - 6$. 52. $9x^2 - 9x - 10$.

Factor.

53. $5x^3 - 20xy^2$. 54. $2ax^2 + 28ax + 98a$.
55. $10x^5 - 80x^4 + 160x^3$. 56. $7x^2 - 35x - 42$.
57. $x^4 - 16y^4$. 58. $r^8 - s^8$.
59. $x^4 - 6x^2 - 27$. 60. $x^4 + 4x^2 - 5$.
61. $-3x^2 + 8x - 4$. 62. $-2x^2 + 3x + 35$.
63. $5t + 6t^2 - 56$. 64. $15 + 4s^2 + 23s$.
65. $75x^2 - 94x - 8$. 66. $18x^2 - 63x + 40$.
67. $-x^2 + 26xy - 169y^2$. 68. $\dfrac{x^2}{y^2} - 2 + \dfrac{y^2}{x^2}$.

Factor each expression without expanding any part of it.

69. $a^2 - 4a(r - s) - 12(r - s)^2$.

Solution. Just as $a^2 - 4ab - 12b^2 = (a - 6b)(a + 2b)$,
so $a^2 - 4a(r - s) - 12(r - s)^2 = [a - 6(r - s)][a + 2(r - s)]$
$= [a - 6r + 6s][a + 2r - 2s]$.

70. $a^2 + 3a(s + t) - 10(s + t)^2$.
71. $(a^2 + 6a)^2 + 14(a^2 + 6a) + 45$.
72. $(a^2 + a)^2 - 8(a^2 + a) + 12$.
73. $(r + s)^2 - (u + v)^2$.
74. $(a - b)^2 - 4(c - d)^2$.
75. $(s + t)^2 + 12(s + t) + 36$.
76. $(x + y)^2 - 20(x + y) + 100$.
77. Explain why $(a^2 - 2ab + b^2)$ is the square of $(b - a)$ as well as of $(a - b)$.

14. Factoring by grouping. Polynomials consisting of four or more terms can sometimes be factored by (1) grouping the terms, (2) factoring within each group, and then (3) applying previously discussed methods to the groups as units.

Illustration 1.

$$\begin{aligned} 2rx + 15y + 6ry + 5x &= (2rx + 6ry) + (5x + 15y) \\ &= 2r(x + 3y) + 5(x + 3y) \\ &= (x + 3y)(2r + 5).^* \end{aligned}$$

Illustration 2.

$$\begin{aligned} x^3 - 3x^2 - 7x + 21 &= (x^3 - 3x^2) - (7x - 21) \\ &= x^2(x - 3) - 7(x - 3) \\ &= (x - 3)(x^2 - 7). \end{aligned}$$

In the foregoing illustrations our objective is to group the terms in such a way that a common factor occurs in each group. In Illustration 2 this common factor is $(x - 3)$.

Illustration 3.

$$\begin{aligned} r^2 + 2st - s^2 - t^2 = r^2 - (s^2 - 2st + t^2) &= r^2 - (s - t)^2 \\ = [r + (s - t)][r - (s - t)] & \\ = [r + s - t][r - s + t]. & \end{aligned}$$

Trinomials of the form $p^2x^4 + qx^2y^2 + r^2y^4$ can sometimes be reduced to the difference of two squares by the addition and subtraction of the same term.

Example 1. Factor $x^4 + 5x^2y^2 + 49y^4$.

Solution. The given expression would be a perfect square if the middle term were $14x^2y^2$.† We shall, accordingly, add $9x^2y^2$ to the $5x^2y^2$ and then neutralize by subtracting $9x^2y^2$ from the expression.

$$\begin{aligned} & x^4 + 5x^2y^2 + 49y^4 \\ &= x^4 + 14x^2y^2 + 49y^4 - 9x^2y^2 \\ &= (x^2 + 7y^2)^2 - (3xy)^2 \\ &= [x^2 + 7y^2 + 3xy][x^2 + 7y^2 - 3xy]. \end{aligned}$$

* This is the same as $2rA + 5A = A(2r + 5)$, where A is $(x + 3y)$.

† Or $-14x^2y^2$.

Exercise 6

Factor by grouping.

1. $5a(x + y) + 6b(x + y)$.
2. $2r(x + 8y) - 3s(x + 8y)$.
3. $ax + bx + ay + by$.
4. $2rx - 3sx + 2ry - 3sy$.
5. $5ar - 10br - 6as + 12bs$.
6. $4az + 7bz - 8aw - 14bw$.
7. $6x^3 - 6x^2 + 7x - 7$.
8. $x^3 - 4x^2 - 2x + 8$.
9. $21x^3 + 14x^2 - 15x - 10$.
10. $x^3 + 6x^2 + 9x + 54$.
11. $w(a - b) + z(b - a)$.
12. $r(p - q) - s(q - p)$.
13. $x^2 - (a^2 - 6ab + 9b^2)$.
14. $r^2 - (a^2 + 10ab + 25b^2)$.
15. $x^2 + 18xy + 81y^2 - r^2$.
16. $s^2 - 4st + 4t^2 - 9w^2$.
17. $121x^2 - a^2 - 16ab - 64b^2$.
18. $100x^2 - c^2 + 14cd - 49d^2$.
19. $36r^2 - y^2 - 1 - 2y$.
20. $16x^2 - 10s - 25 - s^2$.

Reduce to the difference of two squares and then factor.

21. $x^4 + 2x^2 + 81$.
22. $x^4 + x^2 + 25$.
23. $x^4 + 4$.
24. $x^4 + 64$.
25. $x^4 - 34x^2 + 1$.
26. $x^4 - 5x^2y^2 + 100y^4$.
27. $x^4 - 55x^2y^2 + 9y^4$.
28. $x^4 - 3x^2 + 1$.
29. $25x^4 - 106x^2y^2 + 49y^4$.
30. $4x^4 - 25x^2y^2 + 16y^4$.
31. $9x^4 - 28x^2y^2 + 36y^4$.
32. $81x^4 + 4y^4$.

Factor.

33. $a^2 + 2ab + b^2 + a + b$.
34. $25x^2 - 16y^2 - 5x - 4y$.
35. $3r + 2s - 9r^2 + 4s^2$.
36. $6w + 7z - 49z^2 + 36w^2$.
37. $a^2(s - 3t) + b^2(3t - s)$.
38. $x^2(7w - z) + 4y^2(z - 7w)$.
39. $a^2 + 10ab + 25b^2 - x^2 + 18xy - 81y^2$.
40. $a^2 - 6ab + 9b^2 - x^2 - 14xy - 49y^2$.
41. $4r^2 + 9t^2 - x^2 - 16y^2 + 12rt - 8xy$.
42. $36u^2 - z^2 - 25 + v^2 - 12uv + 10z$.
43. $ab(x + y)c + a(x + y)bd - ab(x + y)$.
44. $(x^2 - y^2) - a(x + y)^2 + (x + y)$.
45. $3x + 3y + 3z - ax - ay - az + bx + by + bz$.
46. $2y^2 + 9y - y^3 - 18$.

15. Factoring the sum or difference of two cubes.

[6] $$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
[7] $$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

The student should verify each of these formulas by actually multiplying the two factors on the right side and demonstrating that the resulting product is equal to the left side.

Example 1. Factor $8x^3 + 125$.

Solution. Obviously $8x^3$ is the cube of $2x$ and 125 is the cube of 5. We shall use [6] with a replaced by $2x$ and b replaced by 5.

Just as $$a^3 + b^3 = (a + b)(a^2 - ab + b^2),$$
so $$8x^3 + 125 = (2x + 5)(4x^2 - 10x + 25).$$

Example 2. Factor $x^6 - 64y^3$.

Solution. This is the difference of two cubes; x^6 is the cube of x^2; $64y^3$ is the cube of $4y$.

Using $$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$
we get $$x^6 - 64y^3 = (x^2 - 4y)(x^4 + 4x^2y + 16y^2).$$

16. Factoring the expressions $x^n + y^n$ and $x^n - y^n$. If n is an even number, then $x^n - y^n$ can be considered as the difference of two squares.

Illustration 1.

$$\begin{aligned} x^4 - y^4 &= (x^2)^2 - (y^2)^2 = (x^2 + y^2)(x^2 - y^2) \\ &= (x^2 + y^2)(x + y)(x - y). \end{aligned}$$

If n is a multiple of 3, then $x^n + y^n$ and $x^n - y^n$ can be factored by considering them as the sum or difference of two cubes.

Illustration 2.

$$x^{12} + y^{12} = (x^4)^3 + (y^4)^3 = (x^4 + y^4)(x^8 - x^4y^4 + y^8).$$

If n is odd and not a multiple of 3, then

$$\begin{aligned} x^n + y^n &= (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + y^{n-1}), \\ x^n - y^n &= (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}). \end{aligned}$$

The two foregoing formulas are true for all odd values of n. If, however, n is a multiple of 3, it is better to consider the expression as the sum or difference of two cubes. By so doing, we can easily separate the given expression into more than two factors.

Illustration 3. $x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$.

Exercise 7

Factor.

1. $x^3 + 8$.
2. $x^3 - 8$.
3. $r^3 - s^3$.
4. $w^3 + z^3$.
5. $x^3 - 1000$.
6. $x^3 + 1000$.
7. $s^3 + \frac{1}{27}$.
8. $y^3 - \frac{1}{125}$.
9. $64x^3 + 125$.
10. $27x^3 - 64y^3$.
11. $8x^6 - 1$.
12. $1000x^6 + 27y^3$.
13. $x^9 - y^9$.
14. $x^9 + y^9$.
15. $x^{15} + y^{15}$.
16. $x^{15} - y^{15}$.
17. $(r + s)^3 + 27$.
18. $(p - q)^3 - 125$.
19. $125x^3 - (u - v)^3$.
20. $8x^3 + (u + z)^3$.
21. $2x^4 - 16x$.
22. $81a^2x^2 + 3a^2x^5$.
23. $x^7 + y^7$.
24. $x^7 - y^7$.
25. $x^5 - 32$.
26. $x^5 + 32$.
27. $x^8 - y^8$.
28. $x^6 - y^6$.
29. $x^6 + y^6$.
30. $x^{12} - y^{12}$.

chapter 3 — *Fractions*

17. The degree of a polynomial. We say that a given term is of the nth degree in a certain letter if n is the exponent that appears on that letter. For example, $5x^3y^4$ is of the 3rd degree in x and of the 4th degree in y. The degree of a given term is defined as the sum of the exponents that appear on the letters involved in that term. Thus $5x^3y^4$ is said to be of the 7th degree, or of the 7th degree in x and y. The degree of a polynomial is the same as that of its term of highest degree. For example, $x^5y^3 + y^6 + x^4 + 2xy$ is of the 8th degree, or of the 8th degree in x and y. It is of the 5th degree in x and of the 6th degree in y.

18. Lowest common multiple. A **multiple** of an integer a is any number that results when a is multiplied by another integer. For example, 6, 9, 12, etc., are multiples of 3.

The **lowest common multiple (L.C.M.)** of two or more integers is the smallest number that is a multiple of each of the given integers. For instance, the L.C.M. of 4, 5, and 6 is 60. In other words, 60 is the smallest number that is divisible by 4, 5, and 6.

To find the L.C.M. of two or more algebraic expressions.

(1) *Find the prime factors of each expression.*

(2) *The* L.C.M. *is the product of the different prime factors, with each factor having the highest exponent that it has in any of the given expressions.*

Illustration 1. The L.C.M. of x^2yz^6, xyz, and x^3z^4 is x^3yz^6.

Example 1. Find the L.C.M. of $x^2 - 7x + 6$, $x^2 - 2x + 1$, and $3x^2 - 18x$.

Solution.

$$\begin{aligned} x^2 - 7x + 6 &= (x-1)(x-6) \\ x^2 - 2x + 1 &= (x-1)^2 \\ 3x^2 - 18x &= 3x(x-6) \end{aligned}$$

Hence the L.C.M. is $3x(x-1)^2(x-6)$.

The L.C.M. of two or more integers may be found by using the foregoing method.

Example 2. Find the L.C.M. of 48, 60, and 90.

Solution. After resolving the numbers into their prime factors, we find

$$\begin{aligned} 48 &= 2^4 \cdot 3 \\ 60 &= 2^2 \cdot 3 \cdot 5 \\ 90 &= 2 \cdot 3^2 \cdot 5 \end{aligned}$$

$$\begin{array}{r|l} 2 & 48 \\ \hline 2 & 24 \\ \hline 2 & 12 \\ \hline 2 & 6 \\ \hline & 3 \end{array} \qquad \begin{array}{r|l} 2 & 60 \\ \hline 2 & 30 \\ \hline 3 & 15 \\ \hline & 5 \end{array} \qquad \begin{array}{r|l} 2 & 90 \\ \hline 3 & 45 \\ \hline 3 & 15 \\ \hline & 5 \end{array}$$

The L.C.M. of 48, 60, and 90 is $2^4 \cdot 3^2 \cdot 5 = 720$.

19. Highest common factor. The **highest common factor (H.C.F.)** of two or more integers is the largest number that is a factor of each of the given integers. For example, the H.C.F. of 18, 24, and 60 is 6. The H.C.F. of two or more algebraic expressions is the product of all the factors common to the given expressions.

Illustration 1. The H.C.F. of $6x^7y^8$, $8x^5y^3$, and $12xy^6$ is $2xy^3$.

Exercise 8

Find the L.C.M. and H.C.F. of the following expressions.

1. 30, 45, 75.

2. 12, 28, 42.

3. $9x^2y$, $6xy^3$, $15xy$.

4. $8a^7b^2$, $12a^4b^5$, $32a^3b^6$.

5. $x^2 + 5x + 6$, $x^2 + 4x + 3$, $x^2 + 7x + 12$.

6. $8x^2 - 72$, $2x^2 - 12x + 18$, $6x^2 - 18x$.

Find the L.C.M. of the following expressions. Leave results in factored form.

7. $x^2 - 8x + 16$, $x^2 + 5x - 36$.

8. $2x - 12$, $5x - 30$.

9. $3x + 21, 4x + 28, 6x + 42.$

10. $x^2 + x - 72, x^2 - 64, x^2 - 81.$

11. $x^2 - 2x - 8, x - 4, 3.$

12. $x^2 - 3x - 10, x - 5, 1.$

13. $x^2 - 10x + 25, x^2 - 6x + 5, x^2 - 2x - 15.$

14. $9x^2 - 30x + 25, 6x - 10, 8.$

15. $x^3 + y^3, x^2 - y^2, x + y.$

16. $x^3 - 1, x^3 - x, x^3 - x^2.$

17. $(x - y)^3, x^2 - y^2, x - y.$

18. $x^4 + x^2 + 1, x^3 + 1, x^3 - 1.$

20. The fundamental principle for fractions. *The value of a fraction is not changed if both numerator and denominator are multiplied, or divided, by the same quantity* (*not zero*).

Illustration 1. $\frac{2}{3} = \frac{2 \cdot 5}{3 \cdot 5} = \frac{10}{15}; \frac{6}{14} = \frac{6 \div 2}{14 \div 2} = \frac{3}{7}; \frac{a}{b} = \frac{ax}{bx}; \frac{1}{c} = \frac{m}{cm}.$

21. Changing the sign before a fraction. A minus sign before a fraction means that the numerator is to be considered as multiplied by (-1). For example, $-\frac{a}{b}$ means $\frac{-a}{b}$. Using the fundamental principle, we may multiply the numerator and denominator of $\frac{-a}{b}$ by (-1) and obtain $\frac{a}{-b}$. Hence

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}.$$

We conclude therefore, that *the sign in front of a fraction must be changed provided either the numerator or the denominator is multiplied by* (-1). If *both* numerator and denominator are multiplied by (-1), the value of the fraction remains the same and its sign should not be changed. If the numerator (or denominator) consists of several terms, then in multiplying it by (-1), we must be careful to change the sign of each term.

Illustration 1.

$$-\frac{x^2 - 5x + 7}{x + 8} = \frac{-x^2 + 5x - 7}{x + 8}. \qquad \frac{x^2 + 5}{9 - x^2} = -\frac{x^2 + 5}{x^2 - 9}.$$

If the numerator (or denominator) is in factored form, then we can multiply it by (-1) by changing the sign of any one * of its factors.

Illustration 2. $\dfrac{5}{9-x^2} = \dfrac{5}{(3-x)(3+x)} = -\dfrac{5}{(x-3)(x+3)}$.

Notice that $(x-3)$ is the negative of $(3-x)$ but $(3+x)$ is equal to $(x+3)$. Only the factor $(3-x)$ was multiplied by (-1). This was neutralized by changing the sign before the fraction.

22. Reducing a fraction to lowest terms. *To reduce a fraction to lowest terms,* †

1. *Factor the numerator and denominator.*
2. *Divide both numerator and denominator by all their common factors.*

Illustration 1. $\dfrac{x^2+5x+6}{x^2+7x+10} = \dfrac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+5)} = \dfrac{x+3}{x+5}$.

Illustration 2. $\dfrac{x^2-5x}{x^3-11x^2+30x} = \dfrac{\overset{1}{\cancel{x(x-5)}}}{\cancel{x(x-5)}(x-6)} = \dfrac{1}{x-6}$.

It is to be noted that when top and bottom are divided by $x(x-5)$, the numerator becomes 1, and not 0.

A large part of the difficulty encountered by the student in fractions is due to a promiscuous "cancellation" of common quantities. For example, in the fraction $\dfrac{a+8}{a+2}$, it is incorrect to cancel the a's and get $\dfrac{8}{2} = 4$. In this case a is not a factor of either the numerator or the denominator. If we cancel (strike out) the a's, we are *subtracting* (not dividing) a from the numerator and denominator. This changes the value of the fraction. If, for instance, the value of a is 3, then the original fraction is $\dfrac{3+8}{3+2} = \dfrac{11}{5}$, but the "reduced fraction" (which should equal the original) is 4. The original fraction $\dfrac{a+8}{a+2}$ is in lowest terms.

* More generally, an odd number.
† For brevity, this is sometimes called *reducing the fraction.*

As a second example, let us consider the reducibility of the fraction $\frac{3a + 7}{2a}$. While a is a factor of the denominator, it is not a factor of the entire numerator. It is a factor of only one term of the numerator. Hence it would be incorrect to cancel the a's. The fraction is already in lowest terms.

The beginner can eliminate mistakes of this kind if he discards the thought of cancellation. A foolproof test, which the student should apply in each reduction, is, "Am I dividing the entire numerator and the entire denominator by the same common *factor?*" The number x can be divided out of a fraction if and only if the fraction can be written in the form

$$\frac{x \cdot (\text{rest of numerator})}{x \cdot (\text{rest of denominator})}.$$

Illustration 3. The fraction $\frac{(x + y)(2 + b)}{(x + y)3 + b}$ is in lowest terms. It cannot be reduced because, although $(x + y)$ is a factor of the numerator, it is not a factor of the denominator.

Exercise 9

Reduce to lowest terms.

1. $\frac{84}{132}$.
2. $\frac{90}{315}$.
3. $\frac{33a^8b^6c^2}{44a^2b}$.
4. $\frac{6x^9y}{10x^3y^8}$.
5. $\frac{63a^4b^6}{42a^5b^2}$.
6. $\frac{56a^8b}{20a^2b^2}$.
7. $\frac{6x + 21y}{2x + 7y}$.
8. $\frac{6x^2 + 10}{6y^2 + 10}$.
9. $\frac{4a - 4b}{5a - 5b}$.
10. $\frac{2x + 5}{4x^2 - 25}$.
11. $\frac{(x - 7)^2}{x^2 - 49}$.
12. $\frac{a^3 - a^2b}{a^2 - ab}$.
13. $\frac{x^2 - 12x + 32}{x^2 + 7x - 44}$.
14. $\frac{2x^2 - 6x + 4}{x^2 - 4x + 3}$.
15. $\frac{x^2 - xy}{x^2 - y^2}$.
16. $\frac{3x^2 - 2x - 5}{7x^2 - x - 8}$.
17. $\frac{x + 5}{x^3 + 125}$.
18. $\frac{x^3 - 8}{x^2 - 4}$.

Fill in the missing numerator or denominator.

19. $\dfrac{a}{b} = \dfrac{\quad}{bc}$.

20. $\dfrac{2}{3} = \dfrac{16a}{\quad}$.

21. $\dfrac{a}{b} = \dfrac{a(x+y)}{\quad}$.

22. $\dfrac{x}{x+3} = \dfrac{\quad}{x^2-9}$.

23. $\dfrac{2a-3b}{5c-7d} = \dfrac{\quad}{5ac-7ad}$.

24. $\dfrac{x+6}{a} = \dfrac{\quad}{ax-6a}$.

25. $\dfrac{4}{x+1} = \dfrac{\quad}{x^2+3x+2}$.

26. $\dfrac{278}{1946} = \dfrac{1}{\quad}$.

Rewrite each fraction with no minus sign in either numerator or denominator.

27. $\dfrac{-x}{7}$.

28. $\dfrac{y}{-8}$.

29. $-\dfrac{a}{-b}$.

30. $-\dfrac{a(-b)}{-c}$.

31. $\dfrac{-a-b}{6}$.

32. $\dfrac{(-x)(-y)}{-a}$.

33. $\dfrac{(-4)(-a)}{-b-c}$.

34. $\dfrac{-r-s}{-x-y}$.

Identify as true or false and give reasons.

35. $\dfrac{2-x}{3-y} = \dfrac{x-2}{y-3}$.

36. $-\dfrac{x-9}{y+5} = \dfrac{x+9}{y+5}$.

37. $\dfrac{x+7}{6-y} = -\dfrac{x+7}{y-6}$.

38. $\dfrac{a}{b} = -\dfrac{-a}{b} = -\dfrac{a}{-b} = \dfrac{-a}{-b}$.

39. $\dfrac{(a-b)(b-c)(c-a)}{(c-b)(a-c)(a-b)} = 1$.

40. $\dfrac{(4-x)(5-x)}{6} = \dfrac{(x-4)(x-5)}{6}$.

41. $\dfrac{3a+b}{3c+d} = \dfrac{a+b}{c+d}$.

42. $\dfrac{(1-x)(2-x)(3-x)}{x-4} = -\dfrac{(x-1)(x-2)(x-3)}{x-4}$.

43. $\dfrac{x^2(a-b)}{y^2+(a-b)} = \dfrac{x^2}{y^2+1}$.

44. $\dfrac{(x-y)(a-b)}{(x-y)a-b} = \dfrac{a-b}{a-b} = 1$.

Reduce to lowest terms.

45. $\dfrac{6(a-b)}{2(b-a)}$.

46. $\dfrac{a^2-ab}{(b-a)^2}$.

47. $\dfrac{(y-x)^2}{x^2-y^2}$.

48. $\dfrac{ax-7x}{7y-ay}$.

49. $\dfrac{a^3 - b^3}{bx - ax}$.

50. $\dfrac{x^4 - y^4}{x^6 - y^6}$.

51. $\dfrac{(a + b)^2 + a + b}{a + b}$.

52. $\dfrac{3ax + 4ay + 6bx + 8by}{5ax - 6ay + 10bx - 12by}$.

53. Which of the following fractions are in lowest terms?

(*a*) $\dfrac{x}{x + 1}$. (*b*) $\dfrac{2a}{a + 2}$. (*c*) $\dfrac{a + 3}{a + 6}$.

(*d*) $\dfrac{abx}{ab + bx}$. (*e*) $\dfrac{a + b(x + 1)}{c(x + 1)}$.

23. Addition and subtraction of fractions. The sum (or difference) * of two fractions having the *same* denominator is a fraction whose denominator is the common denominator and whose numerator is the sum (or difference) of the numerators of the given fractions.

Illustration 1.

$$\frac{a}{x} + \frac{b}{x} = \frac{a + b}{x}. \qquad \frac{a}{x} - \frac{b}{x} = \frac{a - b}{x}. \qquad \frac{2}{9} + \frac{4}{9} = \frac{6}{9} = \frac{2}{3}.$$

If a fraction is preceded by a *minus* sign, we must *change the signs of all terms in its numerator* when combining it with other fractions.

Illustration 2.

$$\begin{aligned}\frac{5a + 6b}{7} - \frac{a - 2b + 3}{7} &= \frac{(5a + 6b) - (a - 2b + 3)}{7}\\ &= \frac{5a + 6b - a + 2b - 3}{7}\\ &= \frac{4a + 8b - 3}{7}.\end{aligned}$$

In adding fractions that have different denominators, we must first reduce them to equivalent fractions having a common denominator. This is accomplished by use of the fundamental principle (Art. 20).

The lowest common denominator (L.C.D.) of two or more fractions is the lowest common multiple of their denominators.

* The expression "sum or difference" is called the *algebraic sum.*

Illustration 3. In the following problem, the L.C.D. is 18.

$$2+\frac{1}{3}-\frac{5}{6}+\frac{2}{9}=\frac{36}{18}+\frac{6}{18}-\frac{15}{18}+\frac{4}{18}=\frac{36+6-15+4}{18}=\frac{31}{18}.$$

To perform the indicated addition of several fractions.

1. *Factor each denominator.*
2. *Find the L.C.D.*
3. *For each fraction, multiply the numerator and denominator by an expression * that will convert the given denominator into the required L.C.D.*
4. *Combine these new numerators, placing each numerator in parentheses preceded by the sign of its fraction, and divide by the L.C.D.*

Example 1. Combine into a single fraction:

$$\frac{x}{x^2-10x+25}-\frac{8}{x^2-3x-10}-\frac{1}{x+2}.$$

Solution. Factoring the denominators, we get

$$\frac{x}{(x-5)^2}-\frac{8}{(x-5)(x+2)}-\frac{1}{(x+2)}.$$

The L.C.D. is $(x-5)^2(x+2)$. In order to obtain this expression in each of the three denominators, we multiply top and bottom of the first fraction by $(x+2)$, of the second fraction by $(x-5)$, and of the third fraction by $(x-5)^2$. The fractions then become

$$\frac{x(x+2)}{(x-5)^2(x+2)}-\frac{8(x-5)}{(x-5)^2(x+2)}-\frac{(x-5)^2}{(x-5)^2(x+2)}$$
$$=\frac{(x^2+2x)-(8x-40)-(x^2-10x+25)}{(x-5)^2(x+2)}$$
$$=\frac{x^2+2x-8x+40-x^2+10x-25}{(x-5)^2(x+2)}=\frac{4x+15}{(x-5)^2(x+2)}.$$

Check. Since division by zero is ruled out, the value of the fraction $\dfrac{4x+15}{(x-5)^2(x+2)}$ does not exist when $x=5$ or $x=-2$. For any other

* This expression can usually be found by inspection. It is the quotient of the L.C.D. by the given denominator.

value of x, the algebraic sum of the three given fractions should equal the resulting fraction. For $x = 3$, we have

$$\left.\frac{x}{x^2 - 10x + 25} - \frac{8}{x^2 - 3x - 10} - \frac{1}{x+2}\right]_{x=3} = \frac{3}{4} - \frac{8}{-10} - \frac{1}{5}$$
$$= \frac{15}{20} + \frac{16}{20} - \frac{4}{20} = \frac{27}{20}$$

and
$$\left.\frac{4x + 15}{(x - 5)^2(x + 2)}\right]_{x=3} = \frac{12 + 15}{(-2)^2(5)} = \frac{27}{20}.$$

Example 2. Combine into a single fraction in lowest terms:

$$\frac{7x^2 + 6}{x^2 - 3x} + \frac{2x^2 + 5}{3 - x} + 2x.$$

Solution. Factor the first denominator and write the monomial $2x$ as the fraction $\frac{2x}{1}$:

$$\frac{7x^2 + 6}{x(x - 3)} + \frac{2x^2 + 5}{3 - x} + \frac{2x}{1}.$$

Notice that the second denominator, $3 - x$, is the negative of one of the factors of the first denominator. Multiply the second denominator by (-1) and neutralize by changing the sign of the fraction from $+$ to $-$:

$$\frac{7x^2 + 6}{x(x - 3)} - \frac{2x^2 + 5}{x - 3} + \frac{2x}{1}$$
$$= \frac{(7x^2 + 6) - (2x^2 + 5) \cdot x + 2x \cdot x(x - 3)}{x(x - 3)}$$
$$= \frac{7x^2 + 6 - 2x^3 - 5x + 2x^3 - 6x^2}{x(x - 3)}$$
$$= \frac{x^2 - 5x + 6}{x(x - 3)} = \frac{(x - 2)(x - 3)}{x(x - 3)} = \frac{x - 2}{x}.$$

A partial check can be made by substituting some number for x. Which two numbers must be ruled out?

Exercise 10

Combine into a single fraction in lowest terms. Check if directed by the instructor.

1. $\frac{6x + 4y - 3}{7} - \frac{2x - y + 3}{7}.$ 2. $\frac{5a - 2b + 7}{a + b} - \frac{a - 6b + 7}{a + b}.$

3. $\frac{4x-5}{6}-\frac{3x-8}{6}+\frac{x}{6}$.

4. $\frac{a+7}{x}-\frac{4a+5}{x}+\frac{8a-9}{x}$.

5. $\frac{8(x+3)}{x+2}-\frac{5(x+6)}{x+2}-\frac{2(x-4)}{x+2}$.

6. $\frac{7x+1}{x+1}-\frac{3(x+2)}{x+1}-\frac{x-5}{x+1}$.

7. $\frac{5}{8}+\frac{1}{3}-\frac{1}{6}$.

8. $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}-1$.

9. $-2+\frac{1}{5}-\frac{1}{7}$.

10. $\frac{x-7}{4}+1-\frac{x-2}{6}$.

11. $\frac{a}{9}-2-\frac{5(a+1)}{3}$.

12. $\frac{3(2x-1)}{4}-\frac{x-2}{5}-\frac{7(x+1)}{6}$.

13. $\frac{6x-1}{3}-\frac{4x+1}{5}-\frac{7x-2}{10}$.

14. $\frac{a+3}{a}-\frac{b+1}{ab}-1$.

15. $\frac{x+1}{x^2}-\frac{x+y}{xy}-\frac{x-1}{y}$.

16. $\frac{3x+8}{4x^2}-\frac{2x-1}{x^3}-\frac{5}{8x}$.

17. $\frac{4x-9}{x-2}-\frac{x-1}{2-x}$.

18. $\frac{x}{x+1}-\frac{x+2}{x+3}$.

19. $\frac{7x-1}{8x+40}-\frac{x-1}{6x+30}$.

20. $\frac{8}{x+5}+\frac{7}{5-x}$.

21. $\frac{x}{x-7}-\frac{4}{x+6}-1$.

22. $\frac{9a^2}{3a-1}-3a+1$.

23. $\frac{9x+37}{2x+6}-\frac{5}{x+3}-\frac{1}{2}$.

24. $\frac{a+7}{ax-ay}+\frac{b+9}{by-bx}$.

25. $\frac{5x}{4x+8}-\frac{7}{6x-12}-\frac{8x-9}{3x^2-12}$.

26. $\frac{1}{x^2-5x+6}+\frac{1}{x-2}$.

27. $\frac{1}{x}-\frac{2x-6}{x^2-6x}+\frac{1}{x-6}$.

28. $\frac{8x-9}{x^2-9x}+\frac{7}{9-x}$.

29. $\frac{2x^2-7x-16}{x^2-4x}+\frac{x-7}{4-x}$.

30. $\frac{3x}{x^2+7x+10}-\frac{x+1}{x^2+8x+12}$.

31. $\frac{2x^2}{x^2-25}-\frac{x}{x+5}-1$.

32. $\frac{x}{x-4}-\frac{x+4}{x-1}+\frac{3x-7}{x^2-5x+4}$.

33. $\frac{1}{x^2-25}+\frac{x}{x^2-7x+10}-\frac{6}{x^2+3x-10}$.

34. $\frac{3x}{x^2+5x+4}-\frac{2x}{x^2+7x+12}-\frac{5}{x^2+4x+3}$.

35. $\dfrac{x}{x^2+3x+2}+\dfrac{2}{x^2+4x+3}-\dfrac{2}{x^2+5x+6}.$

36. $\dfrac{x}{x^2-x-2}-\dfrac{1}{x^2+5x-14}-\dfrac{2}{x^2+8x+7}.$

37. $\dfrac{x^2}{x^2+2x+1}+\dfrac{1}{3x+3}-\dfrac{1}{6}.$

38. $\dfrac{x}{x^2-8x+16}-\dfrac{1}{x-4}-1.$

39. $\dfrac{3}{x^2+3x}+\dfrac{2x+5}{x^2+6x+9}-\dfrac{1}{x}.$

40. $\dfrac{x^2+2x+2}{x^2+2x+1}-\dfrac{1}{2x+2}-\dfrac{5}{6}.$

41. $\dfrac{x}{x^2-6x+9}-\dfrac{1}{x-1}-\dfrac{5}{x^2-4x+3}.$

42. $\dfrac{3}{x+3}-\dfrac{2}{3-x}-\dfrac{2x-12}{x^2-9}.$

43. $\dfrac{x+9}{x^2-6x+9}-\dfrac{1}{3-x}.$

44. $\dfrac{x^2-10x+24}{x^2-12x+36}-\dfrac{2}{x-6}-\dfrac{3}{4}.$

45. $\dfrac{7x+9}{8x^3+125}-\dfrac{6}{4x^2-10x+25}.$

46. $\dfrac{8}{(5x-2)^3}+\dfrac{6x}{(5x-2)^2}-\dfrac{1}{5x-2}.$

47. $\dfrac{2x}{(x+1)^2}-\dfrac{7x+4}{x^2-1}+5.$

48. $\dfrac{4x+1}{2x^2-3x+1}+\dfrac{x+5}{3x^2-2x-1}-\dfrac{2}{3x+1}.$

24. Multiplication and division of fractions. *The product of two fractions is equal to the product of their numerators divided by the product of their denominators.*

Illustration 1. $\dfrac{a}{b}\cdot\dfrac{c}{d}=\dfrac{ac}{bd};\quad \dfrac{2}{3}\cdot\dfrac{5}{7}=\dfrac{10}{21}.$

*To divide one fraction by another, invert * the divisor and multiply.*

Illustration 2. $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\dfrac{a}{b}\cdot\dfrac{d}{c}=\dfrac{ad}{bc};\quad \dfrac{5}{6}\div\dfrac{7}{8}=\dfrac{5}{\underset{3}{\cancel{6}}}\cdot\dfrac{\overset{4}{\cancel{8}}}{7}=\dfrac{20}{21}.$

Definition. The **reciprocal** *of* a *is* $\dfrac{1}{a}$. Hence the reciprocal of 4 is $\frac{1}{4}$; the reciprocal of $\dfrac{x}{y}$ is $\dfrac{y}{x}$; the reciprocal of $-\frac{1}{2}$ is -2. We say that $\frac{3}{7}$ and $\frac{7}{3}$ are reciprocals. Hence the reciprocal of a fraction is the frac-

* "Invert" means interchange numerator and denominator.

tion inverted. We now see that the quotient of two fractions is equal to the dividend multiplied by the reciprocal of the divisor.

When multiplying two fractions, first factor their numerators and denominators and then form the product, dividing out all common factors from numerator and denominator.

Illustration 3.

$$\frac{3x^2-8x+5}{x^2+5x-36}\cdot\frac{x^3-4x^2}{6x-10}=\frac{\cancel{(3x-5)}(x-1)}{(x+9)\cancel{(x-4)}}\cdot\frac{x^2\cancel{(x-4)}}{2\cancel{(3x-5)}}=\frac{x^2(x-1)}{2(x+9)}.$$

Check. For $x = 2$, the original product becomes $\dfrac{1}{-22}\cdot\dfrac{-8}{2}=\dfrac{2}{11}$ while the final result becomes $\dfrac{2^2\cdot 1}{2\cdot 11}=\dfrac{2}{11}$.

To multiply a fraction by an integer, multiply the numerator by the integer. To divide a fraction by an integer, multiply the denominator by the integer. In both cases, divide out common factors from numerator and denominator before computing.

Illustration 4.

$$5\left(\frac{1947}{10}\right)=\frac{\cancel{5}\cdot 1947}{\underset{2}{\cancel{10}}}=\frac{1947}{2}$$

$$x\left(\frac{a}{b}\right)=\frac{x}{1}\cdot\frac{a}{b}=\frac{ax}{b}$$

$$\frac{\frac{a}{b}}{x}=\frac{a}{b}\cdot\frac{1}{x}=\frac{a}{bx}.$$

Exercise 11

Perform the indicated operations and express the result in lowest terms. Check if directed by the instructor.

1. $\dfrac{3x}{4y}\cdot\dfrac{2y}{3x}$.
2. $5\left(\dfrac{3}{7}\right)$.
3. $2ab\left(\dfrac{5x}{6a}\right)$.
4. $\dfrac{x}{8}\div 2$.
5. $\dfrac{7}{ab}\div a$.
6. $\dfrac{6}{7}\cdot\dfrac{5}{4}\div\dfrac{45}{14}$.
7. $\dfrac{5a^2b^7}{6x^4}\cdot\dfrac{27x}{15a^3b^5}$.
8. $\dfrac{8a^6}{9b^5}\div\dfrac{4a^2}{15b^4}$.

9. $\dfrac{18ab}{25x} \div \dfrac{9b}{10a}$.

10. $\dfrac{20x^9}{49a^4} \cdot \dfrac{35ay}{48x^3}$.

11. $\dfrac{a-b}{r+s} \cdot \dfrac{s+r}{b-a}$.

12. $\dfrac{mx+2my}{6a-6b} \cdot \dfrac{2a-2b}{bx+2by}$.

13. $\dfrac{2a+6b}{7x-14y} \cdot \dfrac{5x-10y}{10a+30b}$.

14. $\dfrac{4a^2-5a}{a-1} \div \dfrac{16a^2-40a+25}{4a^2-9a+5}$.

15. $\dfrac{x^2+3x-10}{x^2+13x+40} \div \dfrac{x^2-2x}{3x+24}$.

16. $\dfrac{(a-b)^2}{12} \div \dfrac{b-a}{9}$.

17. $\dfrac{x^2-5x-6}{x^2-5x+6} \div \dfrac{x^2+4x-60}{x^2+7x-30}$.

18. $\dfrac{x^2+9x+20}{x^2+7x+10} \cdot \dfrac{x^2+4x+3}{x^2+7x+12}$.

19. $\dfrac{\dfrac{x^2-11x+28}{x^2-16x+63}}{2x-18}$.

20. $\dfrac{\dfrac{x^2-11x-12}{x^2-7x-60}}{x}$.

21. $\dfrac{2x^2+5x-3}{x^2-x-30} \cdot \dfrac{x^2-2x-35}{x^2-4x-21} \div \dfrac{2x^2+3x-2}{x^2-4x-12}$.

22. $\dfrac{x^2-16}{x^2+5x+4} \cdot \dfrac{x^2-7x+10}{x^2-6x+8} \div (x-5)$.

23. $\dfrac{x^2-x-20}{x^2-25} \cdot \dfrac{x^2+5x}{x+1} \div \dfrac{x^2+2x-8}{x^2-x-2}$.

24. $\dfrac{x^2-16}{2x^2+10x+8} \div \dfrac{x^2-13x+36}{x^3+1}$.

25. $\dfrac{x^4+64x}{x-8} \cdot \dfrac{x^2-15x+56}{x^3-3x^2-28x}$.

26. $\dfrac{mr+3ms-2nr-6ns}{rx+3sx-ry-3sy} \cdot \dfrac{(y-x)^3}{mx-2nx-my+2ny}$.

27. State the reciprocal of each of the following quantities:

(*a*) -3 (*b*) $\dfrac{r}{s}$ (*c*) $\dfrac{1}{t}$ (*d*) $m+n$ (*e*) $1+\dfrac{x-y}{x+y}$.

28. What number has no reciprocal?

29. What is the product of a number and its reciprocal?

25. Complex fractions. A fraction is said to be **simple** if it contains no fraction in either its numerator or its denominator. For example $\dfrac{2y}{x^2+3}$ is a simple fraction. A fraction is said to be **complex** if it

contains one or more fractions in either its numerator or denominator, or both.

To simplify a complex fraction, *first reduce the numerator and denominator to simple fractions; then find their quotient.*

Illustration 1.

$$\frac{8-\dfrac{7x^2+17}{x^2-1}}{\dfrac{x}{2}-\dfrac{3x-5}{x-1}}=\frac{\dfrac{8(x^2-1)-(7x^2+17)}{x^2-1}}{\dfrac{x(x-1)-2(3x-5)}{2(x-1)}}=\frac{\dfrac{8x^2-8-7x^2-17}{x^2-1}}{\dfrac{x^2-x-6x+10}{2(x-1)}}$$

$$=\frac{\dfrac{x^2-25}{x^2-1}}{\dfrac{x^2-7x+10}{2(x-1)}}=\frac{(x+5)\cancel{(x-5)}}{(x+1)\cancel{(x-1)}}\cdot\frac{2\cancel{(x-1)}}{\cancel{(x-5)}(x-2)}$$

$$=\frac{2(x+5)}{(x+1)(x-2)}.$$

Check. For $x = 3$, the original complex fraction becomes $\dfrac{8-\frac{80}{8}}{\frac{3}{2}-\frac{4}{2}}=\dfrac{8-10}{-\frac{1}{2}}=(-2)(-2)=4$ while the final result becomes $\dfrac{2(8)}{4(1)}=4$.

Illustration 2. The reciprocal of $\left(1+\dfrac{3}{x}\right)$ is $\dfrac{1}{1+\frac{3}{x}}=\dfrac{1}{\frac{x+3}{x}}=\dfrac{x}{x+3}$.

The importance of indicating clearly the main line of division in certain complex fractions is illustrated by the fact that

$$\frac{\frac{2}{3}}{5}=\frac{2}{15}\qquad\text{while}\qquad\frac{2}{\frac{3}{5}}=\frac{10}{3}.$$

Exercise 12

Simplify. Check if directed by the instructor.

1. $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{8}+\frac{1}{4}+2}$.

2. $\dfrac{\frac{5}{9}-\frac{1}{6}}{4}$.

3. $\dfrac{2}{\frac{7}{8}+\frac{5}{6}}$.

4. $\dfrac{5\frac{1}{3}}{3-\frac{5}{7}}$.

5. $\dfrac{1-\frac{1}{5}}{3\frac{1}{2}}$.

6. $\dfrac{2-\frac{5}{2}}{\frac{1}{4}-\left(-\frac{1}{2}\right)-1}$.

7. $\dfrac{r+s}{\frac{r}{s}+1}$.

8. $\dfrac{x^2-\frac{y^2}{25}}{5x-y}$.

9. $\dfrac{x-\frac{6x-9}{x}}{1-\frac{x+6}{x^2}}$.

10. $\dfrac{1+\frac{8}{b}}{1-\frac{64}{b^2}}$.

11. $\dfrac{\frac{8x}{9y}-\frac{y}{2x}}{\frac{2x+3}{2x}-\frac{y-2}{y}}$.

12. $\dfrac{\frac{a}{b}-\frac{b}{a}}{1-\frac{b}{a}}$.

13. $\dfrac{\frac{x^3}{6}-\frac{1}{6x}}{\frac{x}{3}-\frac{1}{3x}}$.

14. $\dfrac{1+\frac{3}{a}+\frac{9}{a^2}}{a^2-\frac{27}{a}}$.

15. $\dfrac{1-\frac{x-7}{x+7}}{\frac{1}{x+7}-\frac{1}{x-7}}$.

16. $\dfrac{\frac{2r+s}{r+s}-1}{1-\frac{s}{r+s}}$.

17. $\dfrac{\frac{x}{x+1}-\frac{8}{x+6}}{3-\frac{2x+7}{x+1}}$.

18. $\dfrac{\frac{1}{3}-\frac{x-8}{x+2}}{\frac{5}{7}-\frac{x-3}{x+1}}$.

19. $\dfrac{x+3-\frac{x+9}{x+5}}{x+9-\frac{x-6}{x+2}}$.

20. $\dfrac{5-\frac{2}{t+1}}{25+\frac{16}{t^2-1}}$.

21. $\dfrac{1-\frac{6x-2}{x^2-9}}{2-\frac{x+1}{x-3}}$.

22. $\dfrac{1-\frac{x+2}{x^2-2x-8}}{2-\frac{x-1}{x-4}}$.

23. $\dfrac{3-\frac{2x+9}{x+1}}{1-\frac{6x-8}{x^2-x-2}}$.

24. $\dfrac{\frac{x}{4}+\frac{x-20}{x-6}}{3-\frac{2x^2-8}{x^2-36}}$.

25. $\dfrac{\frac{x^2+2x-5}{x^2-25}-1}{7-\frac{4x+5}{x+5}}$.

26. $\dfrac{1-\frac{x+22}{(x+2)^2}}{x+8-\frac{x-2}{x+2}}$.

*27. $1-\dfrac{2}{3-\dfrac{4}{5-\frac{6}{x}}}$.

*28. $1+\dfrac{6}{5+\dfrac{4}{3+\frac{2}{x-1}}}$.

*29. $$\frac{3 - \dfrac{21}{2x+7}}{5x - \dfrac{2x}{1 - \dfrac{3x}{5x+7}}}.$$

*30. $$\frac{\dfrac{8}{x-4} + 1}{2(x^2 - 10) - \dfrac{x^3 + 8}{x + \dfrac{4}{x-2}}}.$$

31. Compute the value of $\dfrac{7x + 8}{2x^2 - x + 4}$ for $x = -\dfrac{3}{2}$.

32. If $x = -\dfrac{5}{2}$, find the value of $\dfrac{3x + 7}{x^2 - x - 9}$.

33. Show that $\left.\dfrac{3x - 1}{5x^2 - 7x + 1}\right]_{x=-\frac{1}{5}} = -\dfrac{8}{13}$.

34. Show that $\left.\dfrac{(2x)^2 + 1}{5x^2 - x - 1}\right]_{x=-\frac{1}{3}} = -13$.

* In these *continued fractions*, begin by simplifying the parts that are farthest from the main line of division.

chapter 4 Linear equations in one unknown

26. Equations. An *equation* is a statement that two expressions are equal. The two expressions are called the **sides** (or **members**) of the equation. There are two kinds of equations: identities (or identical equations) and conditional equations.

An **identity** is an equation that holds true for all permissible * values of the letters involved.

Illustration 1. $x^2 - 25 = (x + 5)(x - 5)$ holds true for all values of x.

Illustration 2. $x^2 + xy - 6y^2 = (x + 3y)(x - 2y)$ holds true for all values of x and y.

Illustration 3. $x - \dfrac{x^2 - 7x}{x - 3} = \dfrac{4x}{x - 3}$ holds true for all permissible values of x, i.e., for all values of x except $x = 3$. When $x = 3$, each side of the equation involves a fraction whose denominator is zero. Such fractions have no meaning and we say their value does not exist.

Illustration 4. The following "trick with numbers" illustrates a simple identity.

Choose any number except 0.
Add 3 to your number.
Square.

* The permissible values of the letters involved are all those values for which each side of the equation has meaning.

Subtract 9.
Divide by your original number.
Subtract your original number.

If you have followed instructions, your result should be 6 regardless of your choice of the original number. To prove this, let x be the original number. Then the numbers that follow are $x + 3$, $(x + 3)^2$ or $x^2 + 6x + 9$, $x^2 + 6x$ or $x(x + 6)$, $x + 6$, and 6. The identity used is

$$\frac{(x + 3)^2 - 9}{x} - x = 6.$$

It holds for all values of x except $x = 0$. Try it for a fraction. For a negative number.

A **conditional equation** is an equation that does not hold true for all permissible values of the letters involved.

Illustration 5. $3x - 1 = 11$ holds true for only one value of x, namely $x = 4$.

Illustration 6. $x^2 - 10x + 21 = 0$ holds true for only two values of x, namely $x = 3$ and $x = 7$.

Illustration 7. $x(x - 5)(x + 8) = 0$ holds true for only three values of x, namely $x = 0$, $x = 5$, and $x = -8$.

Illustration 8. $\dfrac{1}{x + 1} = 0$ holds true for no value of x.

Illustration 9. $2x + y = 5$ is a conditional equation despite the fact that it holds true for infinitely many values of x and y, some of which are $x = 0$ and $y = 5$, $x = 1$ and $y = 3$, $x = -\frac{1}{2}$ and $y = 6$. One set of values for which the equation does not hold true is $x = 10$, $y = 3$.

The difference between an identity and a conditional equation is emphasized by the contrasting definitions:

$$\left\{\begin{matrix}\textit{An identity}\\ \textit{A conditional equation}\end{matrix}\right\} \textit{ is an equation that } \left\{\begin{matrix}\textit{holds true}\\ \textit{does not hold true}\end{matrix}\right\}$$

for all permissible values of the letters involved.

An identity *says* that both sides of an equation are equal for all permissible values. A conditional equation *asks*, "For what values of the unknowns is the left side of this equation equal to the right

side?" The process by which these values are found is called **solving the equation.**

When there is no danger of confusion, the word "equation" is usually used in referring to a conditional equation. When we wish to emphasize the fact that a certain equation is an identity, we sometimes use the symbol "$\equiv$" (read "is identically equal to") instead of the ordinary equality sign.

27. Solutions and roots. An equation is said to be **satisfied** by a set of values of the unknowns if the two sides of the equation become identically equal when these values are substituted for the unknowns. Thus, the equation $2x + 3y = 31$ is satisfied by $x = 5$, $y = 7$.

A **solution of an equation** *is a set of values of the unknowns which satisfies the equation.*

Illustration 1. The equation $2x - y = 7$ has as one solution $x = 5$, $y = 3$. Are there other solutions?

A solution of an equation involving only one unknown is called a **root** of the equation.

Illustration 2. The roots of the equation $3x^2 + x = 10$ are $x = \frac{5}{3}$ and $x = -2$. The student should verify this statement by substitution.

Exercise 13

Identify each of the following equations as an identity or a conditional equation.

1. $x^2 - 7x - 18 = (x - 9)(x + 2)$.
2. $3x - 1 = 5$.
3. $x + y = 4$.
4. $x^3 + 8y^3 = (x + 2y)(x^2 - 2xy + 4y^2)$.
5. Is $\frac{3}{2}$ a root of $2x^2 - 13x + 15 = 0$?
6. Is 0 a root of $7x^2 = 9x$?
7. Is -2 a root of $x^3 - 3x^2 - 4x + 12 = 0$?
8. Is $x = 3$, $y = 2$ a solution of $5x - 7y = 1$?

28. Equivalent equations. Two equations that have exactly the same solutions are said to be **equivalent.**

The following operations always produce equivalent equations.

I. *Adding the same quantity to, or subtracting the same quantity from, both sides.*

II. *Multiplying or dividing both sides by the same quantity provided this quantity is not* 0 *and does not involve any unknown.*

Illustration 1. Given the equation

$$3x - 7 = x + 4. \tag{1}$$

Adding 7 to both sides of (1) gives

$$3x = x + 11. \tag{2}$$

Subtracting x from both sides of (2) gives

$$2x = 11. \tag{3}$$

Dividing both sides of (3) by 2 gives

$$x = \tfrac{11}{2}. \tag{4}$$

Equations (1), (2), (3), and (4) are equivalent. Equation (4) states the solution of the other three equations. The student should verify by substitution that $x = \frac{11}{2}$ is a root of equations (1), (2), and (3).

To **transpose** a term means to move the term from one side of an equation and place it on the other side with its sign changed. This operation is equivalent to subtracting the term from both sides of the equation.

The signs of all terms in an equation may be changed. This amounts to multiplying both sides by -1.

Illustration 2.	$a - x = 4.$
Transpose a:	$-x = 4 - a.$
Change all signs:	$x = a - 4.$

The following operation does not always produce equivalent equations.

III. *Multiplying or dividing both sides by an expression that involves an unknown.*

Illustration 3. In solving the equation $\dfrac{x}{x-1} = \dfrac{2x}{x^2-1} + \dfrac{4}{x+1}$, we shall clear of fractions by multiplying both sides by the L.C.D.,

$x^2 - 1$. This gives $x(x + 1) = 2x + 4(x - 1)$ which becomes $x^2 - 5x + 4 = 0$. The student should verify by substitution that 1 and 4 are roots of the last equation. If $x = 1$ is substituted in the original equation, we get $\frac{1}{0} = \frac{2}{0} + \frac{4}{2}$. Since division by zero is ruled out, we see that $x = 1$ is not a root of the given equation; it was introduced by the multiplication. Although it is a true root of the derived equation, it is said to be an *extraneous root* (i.e., not a root at all) of the original equation. It can be shown by substitution that 4 is a root of both the original and derived equations.

An **extraneous root** *is a value of the unknown that satisfies a derived equation but not the original one.* Whenever a multiplication of type III is performed, all results should be checked by substitution to see if any should be rejected as extraneous.

Illustration 4. The equation $x^2 = 6x$ has the roots $x = 0$ and $x = 6$. If both sides are divided by x, we get $x = 6$. The root $x = 0$ was lost by the division. A division of type III should never be performed.

29. Linear equations. A **linear** (or first degree) **equation in** x is an equation that can be written* in the form $ax + b = 0$, where $a \neq 0$. The following are examples of linear equations in x: $\frac{2}{3}x - \frac{4}{5} = 0$ and $5x - 1 = 3x - 1$. What are the values of a and b in each equation?

To solve a linear equation in one unknown.

1. *Clear of fractions (if they appear) by multiplying both sides by their L.C.D.*

2. *Transpose all terms involving the unknown to one side and all other terms to the other side of the equation.*

3. *Combine like terms and exhibit the unknown as a factor of one side.*

4. *Divide both sides by the coefficient of the unknown.*

To check the solution, substitute the value of the unknown in the *original* equation.

Example 1. Solve for x: $\dfrac{x}{4} - \dfrac{2x - 1}{6} = 1 - \dfrac{x + 2}{3}$.

* By performing operations I and II.

Solution. The L.C.D. is 12. Clear of fractions by multiplying both sides by 12.

$$12\left(\frac{x}{4}\right) - 12\left(\frac{2x-1}{6}\right) = 12 \cdot 1 - 12\left(\frac{x+2}{3}\right)^*$$

$$3x - 2(2x - 1) = 12 - 4(x + 2)$$
$$3x - \cancel{4x} + 2 = 12 - \cancel{4x} - 8$$
$$3x = 12 - 8 - 2$$
$$x = \tfrac{2}{3}.$$

Check. Set $x = \frac{2}{3}$ in the original equation.

$$\begin{array}{l|l} \dfrac{\frac{2}{3}}{4} - \dfrac{\frac{4}{3} - 1}{6} & \quad 1 - \dfrac{\frac{2}{3} + 2}{3} \\ = \frac{1}{6} - \frac{1}{18} & \quad = 1 - \frac{8}{9} \\ = \frac{1}{9} & \quad = \frac{1}{9} \end{array}$$

True

Since we performed no multiplication of type III, the check is for error only.

Example 2. Solve for x: $\dfrac{5}{x-4} - \dfrac{3}{x-1} - \dfrac{x+11}{x^2-5x+4} = 0.$

Solution. Multiply both sides by the L.C.D., $(x - 1)(x - 4)$.

$$5(x - 1) - 3(x - 4) - (x + 11) = 0$$
$$5x - 5 - 3x + 12 - x - 11 = 0$$
$$5x - 3x - x = 5 - 12 + 11\dagger$$
$$x = 4.$$

Check. Substituting $x = 4$ in the original equation, we find that the first fraction becomes $\dfrac{5}{4-4} = \dfrac{5}{0} =$ impossible. Since division by 0 is ruled out, $x = 4$ is an extraneous root of the given equation. The original equation has no solution (or no root). What operation introduced the extraneous root?

An equation that involves more than one letter is called a **literal** equation (or a formula). Any one of the letters may be considered as the unknown.

* This step should be done mentally.

† After a little practice, the student can omit this step.

Example 3. Solve for x: $\dfrac{2x - a}{x + 6} = b.$

Solution. Clearing of fractions we get

$$2x - a = b(x + 6)$$
$$2x - a = bx + 6b$$

Isolate terms involving x: $\quad 2x - bx = a + 6b$

Factor the left side: $\quad x(2 - b) = a + 6b$

Divide both sides by $(2 - b)$: $\quad x = \dfrac{a + 6b}{2 - b}.$

Check. The student should show that the original equation is satisfied when x is replaced by $\dfrac{a + 6b}{2 - b}$. Although a multiplication of type III was performed, the original equation and its derived equations are, in this case, equivalent.

Exercise 14

Solve each equation and check the solution.

1. $7x + 3 = 4x + 5.$

2. $2x + 3 = x - 6.$

3. $3x + 4 = 7x + 6.$

4. $6x - 5 = 1 - 2x.$

5. $8 + 3x = -4(x - 2).$

6. $3(5 - y) = -4(6y + 5).$

7. $8 - \dfrac{x}{2} = 0.$

8. $\dfrac{7}{x} - 1 = 0.$

9. $4 = \dfrac{5}{x}.$

10. $6 = \dfrac{x}{3}.$

11. $4x - \frac{1}{5} = 3x + \frac{1}{10}.$

12. $6x + \frac{1}{2} = 2x + \frac{1}{8}.$

13. $13x - \frac{2}{3} = 8(\frac{3}{4} + x).$

14. $4 + x - 6(\frac{2}{3} - x) = 0.$

15. $5t - .79 = 2t - .4.$

16. $3x - .17 = 8x + .63.$

17. $.9x + .15 = .7x - .33.$

18. $.11t + 2.89 = 1.77 + 3.47t.$

19. $\dfrac{x + 9}{2} - \dfrac{5x + 3}{6} = 2.$

20. $\dfrac{3x - 2}{5} - \dfrac{x + 3}{7} = 1.$

21. $\dfrac{x - 7}{2} - \dfrac{x - 1}{8} = \dfrac{3}{4}.$

22. $\dfrac{7x - 2}{3} - \dfrac{4x - 1}{5} = \dfrac{3(1 - x)}{5}.$

23. $\dfrac{2y + 7}{3} - \dfrac{y - 5}{4} = \dfrac{7y - 1}{6}.$

24. $\dfrac{x + 1}{9} - \dfrac{x + 5}{6} = \dfrac{x + 3}{2}.$

25. $\dfrac{1}{2x} - \dfrac{1}{3x} - \dfrac{1}{4x} = \dfrac{1}{6}.$

26. $(2x - 3)(2x + 1) - (4x + 7)(x - 2) = 13.$

27. $\dfrac{4x^2 + 7}{6x} - \dfrac{2x + 5}{3} = \dfrac{1}{x}.$

28. $\dfrac{8x - 1}{7x} - \dfrac{x - 1}{3x} = 1.$

29. $\dfrac{x^2}{x - 4} - \dfrac{4}{7} - x = 0.$

30. $\dfrac{x - 3}{x - 7} = 2.$

31. $\dfrac{1}{2(x - 3)} - \dfrac{x + 5}{x - 3} = \dfrac{1}{2}.$

32. $\dfrac{x}{x + 1} + \dfrac{x}{x + 3} = 2.$

33. $\dfrac{10x^2}{5x - 1} - 2x - 1 = 0.$

34. $\dfrac{x}{x + 2} - \dfrac{x - 2}{3(x + 1)} - \dfrac{2}{3} = 0.$

35. $\dfrac{6x - 4}{x - 7} = 2 + \dfrac{x - 8}{7 - x}.$

36. $\dfrac{x}{x - 4} = \dfrac{x^2 + 20}{x^2 - 8x + 16}.$

37. $\dfrac{2x}{3x + 3} - \dfrac{x + 2}{6x + 6} - \dfrac{x - 6}{8x + 8} = \dfrac{5}{12}.$

38. $\dfrac{7}{x - 5} = \dfrac{x^2 - 10}{x^2 - x - 20} - 1.$

39. $\dfrac{x}{2x - 1} = \dfrac{4x^2 - 7x + 60}{8x^2 - 6x + 1}.$

40. $\dfrac{x}{x + 1} - \dfrac{x + 2}{x + 6} = \dfrac{25}{x^2 + 7x + 6}.$

41. $\dfrac{9}{x - 4} - \dfrac{5}{x + 1} = \dfrac{27}{x^2 - 3x - 4}.$

42. $\dfrac{5x - 22}{x^2 - 6x + 9} - \dfrac{11}{x^2 - 3x} - \dfrac{5}{x} = 0.$

43. $\dfrac{x - 12}{x^2 - 10x + 25} - \dfrac{3}{x^2 - 5x} - \dfrac{1}{x} = 0.$

44. $\dfrac{2x + 1}{5x - 2} + \dfrac{x + 7}{3x - 4} - \dfrac{11x^2 + 8x - 3}{15x^2 - 26x + 8} = 0.$

45. $\dfrac{x}{x - 8} - \dfrac{5}{x + 8} - \dfrac{x^2 + 64}{x^2 - 64} = 0.$

46. $\dfrac{7}{x - 5} - \dfrac{6}{x + 3} = \dfrac{48}{x^2 - 2x - 15}.$

47. $\dfrac{x + 3}{x} - \dfrac{x + 4}{x + 5} = \dfrac{15}{x^2 + 5x}.$

48. $\dfrac{6x-1}{5x+3}+\dfrac{x}{x-2}-\dfrac{11x^2+8}{5x^2-7x-6}=0.$

49. $\dfrac{x+5}{2x-8}-\dfrac{x+4}{6x-30}-\dfrac{1}{3}=0.$

50. $\dfrac{3x^2}{27x^3+8}+\dfrac{5x-4}{90x^2-60x+40}-\dfrac{1}{6x+4}=0.$

Solve for x in terms of the other letters involved. Check if so directed by the instructor.

51. $7x+b=a.$

52. $ax-3=b.$

53. $ax+x=6.$

54. $ax=x+b.$

55. $5x=a+bx.$

56. $x^2+a^2=(x-a)^2.$

57. $ax+1=x+a^2.$

58. $bx+6a=2ab+3x.$

59. $\dfrac{a}{b}=\dfrac{c}{x}.$

60. $\dfrac{2a}{b}=\dfrac{x}{2d}.$

61. $a=\dfrac{x}{b}.$

62. $a=\dfrac{b}{x}.$

63. $\dfrac{ax-b}{x-2}=c.$

64. $\dfrac{x+6}{2x+b}=a.$

65. $a=\dfrac{x-9}{3x-b}.$

66. $a=\dfrac{b-x}{1+bx}.$

67. $\dfrac{a-x}{b-x}-1=c.$

68. $ax=\dfrac{5bx+ax^2+c-x}{x}.$

69. $\dfrac{4}{a}-\dfrac{1}{x}=\dfrac{1}{x}-\dfrac{3}{b}.$

70. $\dfrac{x}{ab}-\dfrac{x-a}{b^2}-\dfrac{x-b}{a^2}=0.$

71. Solve for g: $s=\frac{1}{2}gt^2.$

72. Solve for h: $S=\pi r(r+2h).$

73. Solve for C: $F=\frac{9}{5}C+32.$

74. Solve for r: $S=\dfrac{a-rl}{1-r}.$

75. Solve for x in terms of y: $4x+5xy+6y=7.$

76. Solve for x in terms of y: $x+y=1+xy.$

77. Solve for y in terms of x: $3x+xy-5y+6=0.$

78. Solve for y in terms of x: $(3x+y)^2=(x+y)^2.$

79. Solve $A=P+Prt$: (1) for P; (2) for t.

80. Solve $l=a+(n-1)d$: (1) for d; (2) for n.

81. Solve $S=\dfrac{n}{2}(a+l)$: (1) for n; (2) for a.

82. Solve $A = \frac{1}{2}h(b + B)$: (1) for b; (2) for h.

83. Solve $\dfrac{1}{f} = \dfrac{1}{a} + \dfrac{1}{b}$: (1) for f; (2) for a.

84. Solve $x = a + k(b - a)$: (1) for a; (2) for b.

30. Stated problems. Up to this point, we have confined our discussion to formal processes, such as factoring, simplifying, solving equations, etc. Our sole justification for studying these various procedures is to enable us to solve stated problems.

When a problem is stated verbally, it can frequently be solved by translating the stated conditions into an equation and then solving this equation. The following procedure is recommended.

1. Read the problem slowly and note carefully the various conditions, stated or implied.

2. If there is only one unknown, represent it by a letter. Be specific in designating the unknown. It always represents a number. If there are several unknowns, represent one unknown by a letter and then express the other unknowns in terms of this letter.

3. Translate the conditions of the problem into an equation, i.e., try to find two quantities that are equal.

4. Solve the equation and check the results.

Example 1. Find the two acute angles of a right triangle if the larger is 6 degrees more than twice the smaller.

Solution. Let $x =$ number of degrees in the smaller angle.
Then $90 - x =$ number of degrees in the larger angle.
And $90 - x = 6 + 2x$.

$$84 = 3x; \quad x = 28; \quad 90 - x = 62.$$

Hence the angles are 28° and 62°.

Check. $62° = 6° + 2 \cdot 28°$. True.

Could the problem have been worked by letting x represent the larger angle?

Example 2. How many gallons of a 40% alcohol solution should be added to 9 gallons of a 20% alcohol solution to produce a 28% solution?

Solution. Let $x =$ number of gallons of 40% solution. The actual alcohol content of these x gallons is $.40x$. The alcohol content of

9 gallons of 20% solution is .20(9). The final mixture contains $(x + 9)$ gallons of a 28% solution. Its alcohol content is $.28(x + 9)$. Our equation develops from the fact that the alcohol content of the two ingredients must equal that of the mixture:

$$.40x + .20(9) = .28(x + 9).$$

Multiply by 100: $$40x + 180 = 28x + 252$$

$$12x = 72; \quad x = 6 \text{ (gallons)}.$$

Check. $$.40(6) + .20(9) = .28(15). \quad \text{True.}$$

If a body moves at a constant rate r for a period of time t, then the distance d it travels is given by the formula

$$d = rt.$$

From this equation, $r = \frac{d}{t}$ and $t = \frac{d}{r}$.

Illustration 1. If an auto moving with constant speed travels 200 miles in 5 hours, its speed is

$$r = \frac{d}{t} = \frac{200}{5} = 40 \text{ miles per hour.}$$

Illustration 2. If an airplane travels 400 miles with a speed of 150 miles per hour, the time required is

$$t = \frac{d}{r} = \frac{400}{150} = 2\tfrac{2}{3} \text{ hours.}$$

Example 3. An airplane has an airspeed (speed in still air) of 125 mph.* A west wind of 25 mph is blowing. The plane is to patrol due east and then return to its base. How far east can it go if the round trip is to consume 4 hours?

Solution. When flying east, the plane has a ground speed (actual speed) of $125 + 25 = 150$ mph. When flying west, its ground speed is $125 - 25 = 100$ mph.

Let x = number of miles plane should fly east. Then the time required to go east is $t = \frac{d}{r} = \frac{x}{150}$ and the time going west is $\frac{x}{100}$.

* Miles per hour.

Since the time to go east plus the time to go west must be equal to 4,

$$\frac{x}{150} + \frac{x}{100} = 4.$$

$$2x + 3x = 1200; \quad x = 240 \text{ miles.}$$

Check. $\frac{240}{150} + \frac{240}{100} = 4.$ True.

Exercise 15

1. The difference of the squares of two consecutive integers is 43. Find the integers.

2. Find four consecutive integers having a sum of 142.

3. What number must be added to both numerator and denominator of $\frac{11}{14}$ to produce $\frac{8}{9}$?

4. Divide 20 into two parts such that the square of the larger exceeds the square of the smaller by 280.

5. A vending machine contains $4.30 consisting of nickels and dimes. How many of each are there if the total number of coins is 60?

6. The weekly pay roll of a certain company employing 70 persons is $2860. Some of the employees earn $40 per week. The others earn $45 per week. How many draw each wage?

7. In a certain golf tournament the winner received twice as much money as the runner-up, who, in turn, received twice as much as the person who finished third. What were the three prizes if their total value was $315?

8. The sum of the reciprocals of two consecutive integers is equal to 7 divided by their product. Find the integers.

9. A man has $19.25 in nickels, dimes, and quarters. He has twice as many dimes as nickels and three times as many quarters as dimes. How many coins of each kind does he have?

10. The perimeter of a right triangle is 84 feet. One leg is three-fourths as long as the other. Find the three sides.

11. A grocer has some tea worth 80¢ per pound and some worth 50¢. How many pounds of each should be used in forming 100 pounds of a mixture worth 59¢ per pound?

12. How many pounds of 50¢ coffee should be mixed with 100 pounds of 32¢ coffee to make a blend to sell at 40¢ per pound?

13. How many ounces of 24 carat gold (pure gold) should be added to 5 ounces of 18 carat gold (gold content is 18 parts out of 24 parts) to form 22 carat gold?

14. How much pure silver must be added to 30 pounds of 60% silver and 85 pounds of 40% silver to produce an alloy that is 58% silver?

15. How many pounds of cream having a butterfat content of 40% should be added to 100 pounds of milk in order to raise its butterfat content from 3% to 4%?

16. How many gallons of water must be evaporated from 80 gallons of a 3% salt solution to produce a 5% salt solution?

17. A radiator having a capacity of 16 quarts is filled with a 20% alcohol solution. How much of the solution should be drained out and replaced with pure alcohol if the radiator is to have a 50% solution?

18. A druggist has pure alcohol and a 20% alcohol solution. How many gallons of each should he use in forming b gallons of a 25% solution?

19. A trip of 875 miles requires 7 hours. Part of the trip is by plane at 165 mph; the remainder is by train at 45 mph. How many hours are spent in the plane?

20. A man can row a boat 2 mph in still water. If he can row downstream twice as fast as he can row upstream, what is the speed of the current?

21. A river flows 5 mph. A launch has a speed of 20 mph in still water. How far downstream can the launch go and return if the round trip must be made in 2 hours?

22. An airplane traveled from A to B at 200 mph and then back to A at 300 mph. The round trip took 1 hour. How far is B from A?

23. The speed of a motorboat is 16 mph. Find the speed of the current of a river on which the motorboat can go 3 miles upstream in the same time that it takes to go 5 miles downstream.

24. The current of a river flows 2 mph. Find the speed of a motorboat that goes 8 miles upstream in the same time that it goes 11 miles downstream.

25. The wind is blowing with a speed of 30 mph. An airplane can go 100 miles with the wind in half the time it takes to go 140 miles against the wind. Find the speed of the plane in still air.

26. An airplane travels a hours at a speed of b miles per hour. An auto requires 40 minutes longer to travel the same distance. Find the speed of the auto.

27. A boy weighing 85 pounds and a girl weighing 75 pounds sit at opposite ends of a teeter board 14 feet long. Where should the fulcrum be placed so that the board will balance?
Hint. The weight on one side multiplied by its lever arm (distance to the fulcrum) must equal the weight on the other side times its lever arm.

28. A 52-pound weight is placed on a lever 8 feet from the fulcrum. A 34-pound weight is placed 5 feet from the fulcrum on the other side. Where should a 30-pound weight be placed to balance the lever?

29. A father's age is 40 years when his son's age is 9 years. When will the father be twice as old as his son?

30. A father is three times as old as his son. Eight years ago the father's age was five times that of his son. Find their present ages.

31. A certain task can be performed by A in 4 hours and by B in 6 hours. If they work together, how long does it take them to perform the task?
Hint. What fractional part of the task does A perform in 1 hour? In x hours? The sum of the fractional parts done by A and B in x hours must equal 1.

32. A certain pipe can fill a tank in 3 hours. Another pipe can empty the tank in 4 hours. How long does it take to fill the tank when both pipes are open?

33. A man invests \$1000, parts of it at 3% and the remainder at 5%. The total annual interest is \$33. Find the two sums invested.

34. One-half of a man's principal is invested at 3%. One-third is invested at 4%. The remainder draws 5% interest. Find his total principal if his yearly interest is \$484.

35. For a single man, all income in excess of \$500 is taxed 19% and all income above \$2500 is taxed an additional 4%. What is the income of a single man who pays a tax of \$771?

36. What rate of interest does a man's principal bring if he has a dollars invested at 3% and b dollars invested at 4%?

37. At what time between 3 and 4 o'clock are the hands of a watch in conjunction?

Hint. In x minutes the hour hand moves $\frac{x}{12}$ minute spaces.

38. At what time between 1 and 2 o'clock do the hands of a watch form a straight line?

39. The hands of a clock indicate that it is between 4 and 5 o'clock. After the elapse of slightly less than an hour, the positions of the hands will be interchanged. What time is it now?

40. Find two integers, whose sum is 115, such that when the larger is divided by the smaller, the quotient is 7 and the remainder is 11.
Hint. Recall formula 1, Art. 10.

chapter 5 *Functions and their graphs*

31. Constants and variables. A **constant** is a quantity that does not change in value during the discussion of a given problem. A **variable** is a quantity that may change in value in the course of a certain problem. For example, if a body falls from rest, the distance s it has fallen in the time t is given by

$$s = \tfrac{1}{2}gt^2.$$

If s is in feet and t is in seconds, then g is approximately 32. The quantities $\frac{1}{2}$ and g are constants; but s and t are variables because they change in value as the body falls.

32. Functions. *A* **function** *of* **x** *is a quantity whose value can be determined whenever a value is assigned to* **x**. For example, $5x + 1$ is a function of x. If we assign to x the value 2, then $5x + 1$ takes on the value 11. If $x = -3$, then $5x + 1 = -14$. The value of the quantity $5x + 1$ depends solely on the value of x.

Other examples of functions of x are $x^2 + 3$, $\sqrt{x}$, $2\pi x$, 10^x, and $\dfrac{x+5}{x+6}$. Stated rather loosely, a function of x is any expression that involves no variable other than x. We can say that the number 7 is a function of x. Why? Additional examples of functions are: $(s + 1)^2$ is a function of s; $\dfrac{c}{v}$ is a function of v, provided c is a constant; πr^2 is a function of r, i.e., the area of a circle is a function of its radius.

Let us use the letter y to designate a certain function of x. If this function happens to be $x^2 + 3$, then

$$y = x^2 + 3.$$

With such a relation in mind, we can say that y is a function of x. In general, a variable y is a function of a variable x if for every value of x there corresponds one or more values of y. If y is a function of x, we call x the **independent variable** and y the **dependent variable.**

33. Functional notation. Instead of using a single letter, such as y, to represent a function of x, we frequently use symbols like $f(x)$, $g(x)$, $F(x)$, etc. The letter inside the parentheses is the independent variable; the letter in front of the parentheses is the name of the function. The symbol $f(x)$ is to be read "the f-function of x" or merely "f of x." For example, $f(x)$ might represent $x^2 + 1$ while $g(x)$ could mean $4x + 5$ and $h(x)$ might stand for some undefined function of x. The equation

$$F(t) = t^2 + 3t$$

should be read "the F-function of t is $t^2 + 3t$" or "F of t equals $t^2 + 3t$."

If $f(x)$ represents a certain function of x, and a is any quantity whatsoever, then

$f(a)$ means the value of $f(x)$ when $x = a$.

Illustration 1.

If
$$f(x) = x^2 + 1,$$
then
$$\begin{aligned} f(a) &= a^2 + 1 \\ f(3) &= 3^2 + 1 = 10 \\ f(r+4) &= (r+4)^2 + 1 = r^2 + 8r + 17 \\ f(s^2) &= (s^2)^2 + 1 = s^4 + 1 \\ [f(s)]^2 &= [s^2 + 1]^2 = s^4 + 2s^2 + 1. \end{aligned}$$

The symbol $f(x, y)$ designates a function of the two independent variables x and y. It should be read, "f of x and y." If $f(x, y) = x^2 + y^3 + 1$, then $f(5, 10) = 5^2 + 10^3 + 1 = 25 + 1000 + 1 = 1026$. A similar notation is used for functions of more than two variables

Comments. 1. The student should realize that $f(x)$ does not mean "f times x." It should be read and thought of as "f of x."

2. The symbols $f(2)$, $f(3)$, etc., have no meaning unless $f(x)$ has been defined.

3. If several functions of x occur in a given problem, they must be represented by different letters such as $f(x)$, $g(x)$, etc.

4. The quantity $2x + 5$ is a function of x. The expression $2u + 5$ is the *same* function of u. In either case the independent variable is doubled and added to 5.

Exercise 16

Express in words.

1. $s = f(t)$. **2.** $y = g(x)$. **3.** $h(r)$. **4.** $F(w)$.

Copy and fill in the blanks.

5. (*a*) $\dfrac{s}{s^2 + 2}$ is a function of ....

(*b*) $2^t + t^2$ is a function of ....

6. The cost of 10 gallons of gasoline is a function of ....

7. The perimeter of a square is a function of ....

8. The circumference of a circle is a function of its ....

9. The area of a rectangle with base 10 is a function of its ....

10. The area of a triangle is a function of its .... and ....

Write an equation (a formula) to express each of the following functional relationships.

11. The number of dollars D earned by a worker in x hours at 70 cents per hour.

12. The volume V of a box with length 5, width 3, and height h.

13. The value V in cents of 6 dimes and n nickels.

14. The annual interest I on x dollars at 3%.

15. The freight F (in dollars) on an automobile weighing x pounds if there is a minimum charge of \$4.00 for the first 100 pounds and 3 cents for each additional pound.

16. The hypotenuse z of a right triangle whose legs are 7 and x.

Given $f(x) = 5x + 2$ *and* $g(t) = 3t^2 - 4t$. *Find the following.*

17. $f(3)$. **18.** $f(-10)$. **19.** $f(0)$.
20. $f(\frac{1}{2})$. **21.** $g(-2)$. **22.** $g(a)$.
23. $g(-\frac{1}{3})$. **24.** $6g(5)$. **25.** $[f(w)]^2$.
26. $f(w^2)$. **27.** $\dfrac{f(2)}{f(3)}$. **28.** $f(\frac{2}{3})$.
29. $f(r)g(2)$. **30.** $f(g(z))$.

31. Given $f(t) = \dfrac{t-3}{2t-5}$. Find $7f(6)$, $[f(0)]^3$, $f\left(\dfrac{a}{2b}\right)$.

32. Given $f(x) = 7x^2 - 8x + 9$. Find $\dfrac{f(x+h) - f(x)}{h}$.

33. Given $2x + y - \pi x - 7 = 0$.
(*a*) Express y as a function of x.
(*b*) Express x as a function of y.

34. Given $f(x, y) = x^3 + 2xy + y^2 - 5$. Find $f(1, 4)$, $f(0, 3)$, $f(-2, 1)$.

34. The rectangular coordinate system. A directed line is a line upon which one direction is considered positive; the other, negative.

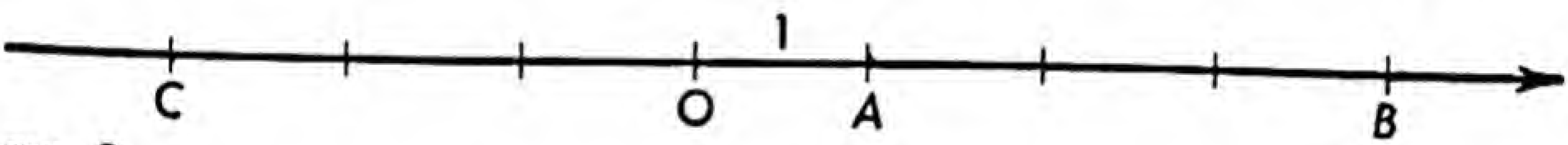


FIG. 2

The arrowhead in Fig. 2 indicates that all segments measured from left to right are positive. If $OA = 1$ unit of length, then $OB = 4$ and $BA = -3$. Observe that since the line is directed, $CB = 7$, whereas $BC = -7$.

A rectangular (or Cartesian) coordinate system consists of two perpendicular *directed* lines. It is customary to draw and direct these lines as in Fig. 3. The **x-axis** and **y-axis** are called the **coordinate axes;** their intersection O is called the **origin.** The position of any point in the plane is fixed by its distances from the coordinate axes.

The x-coordinate * (or x) *of any point P is the directed segment NP* (or *OM*) *measured from the y-axis to point P. The y-coordinate* * (or y) *of point P is the directed segment MP measured from the x-axis*

* The x-coordinate and y-coordinate are also called the abscissa and ordinate, respectively.

to point P. It is necessary to remember that each coordinate is measured *from axis to point*. Thus the x of P is NP (not PN); the y of P is MP (not PM). The point P with x-coordinate x and y-coordinate y is denoted by $P(x, y)$. It follows that the x of any point to the right of the y-axis is positive; to the left, negative. Also the y of any point above the x-axis is positive; below, negative.

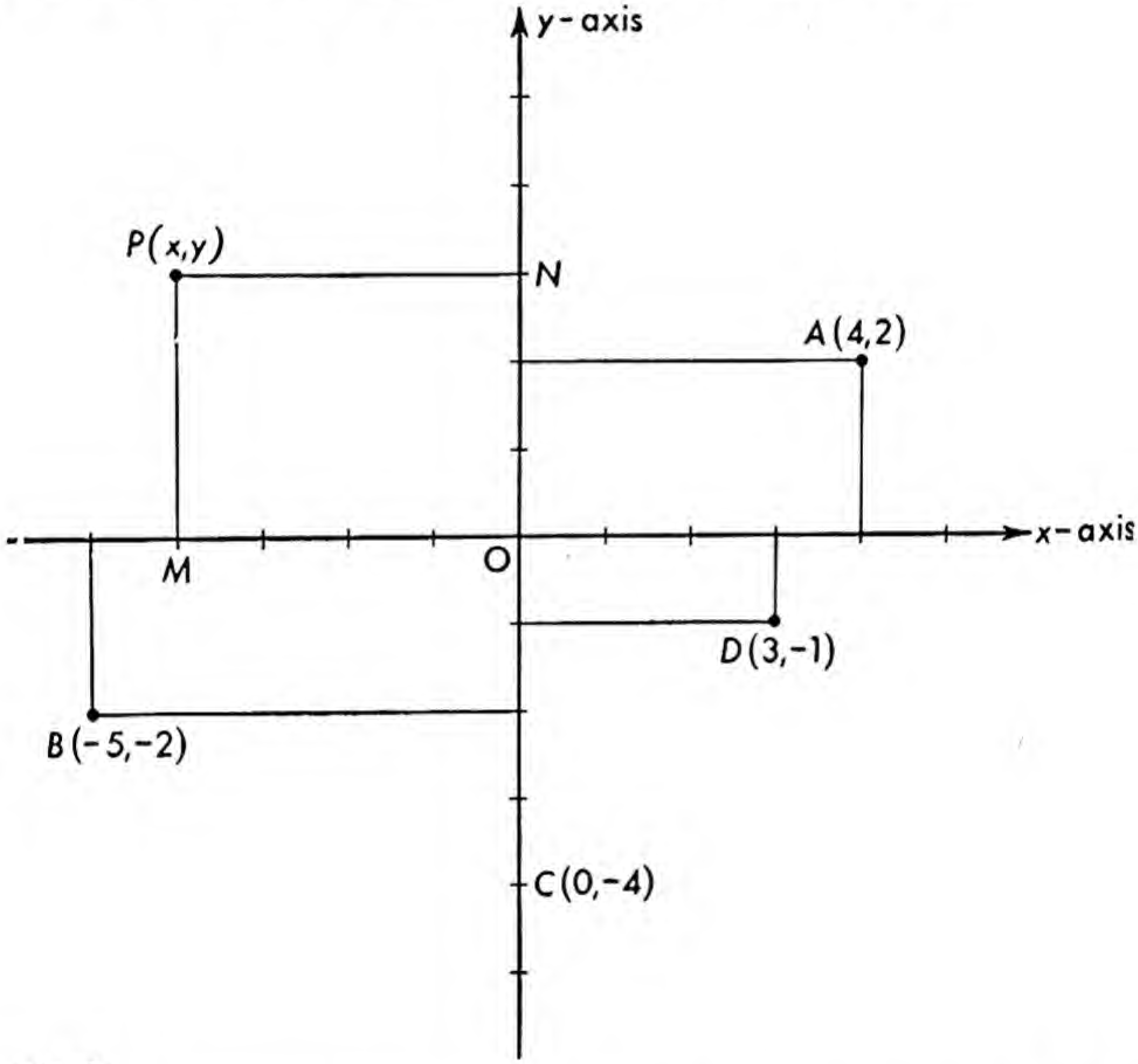


FIG. 3

To **plot** a point means to locate and indicate its position on a coordinate system. Several points are plotted in Fig. 3.

The coordinate axes divide the plane into four parts called **quadrants** as indicated in Fig. 4.

Comments. 1. To each pair of coordinates (x, y) there corresponds one and only one point. Likewise, to each point there corresponds one and only one pair of coordinates.

2. Although we usually use the same unit of length for the scales on the x and y axes, sometimes it is desirable to choose a smaller (or larger) unit on the y-axis.

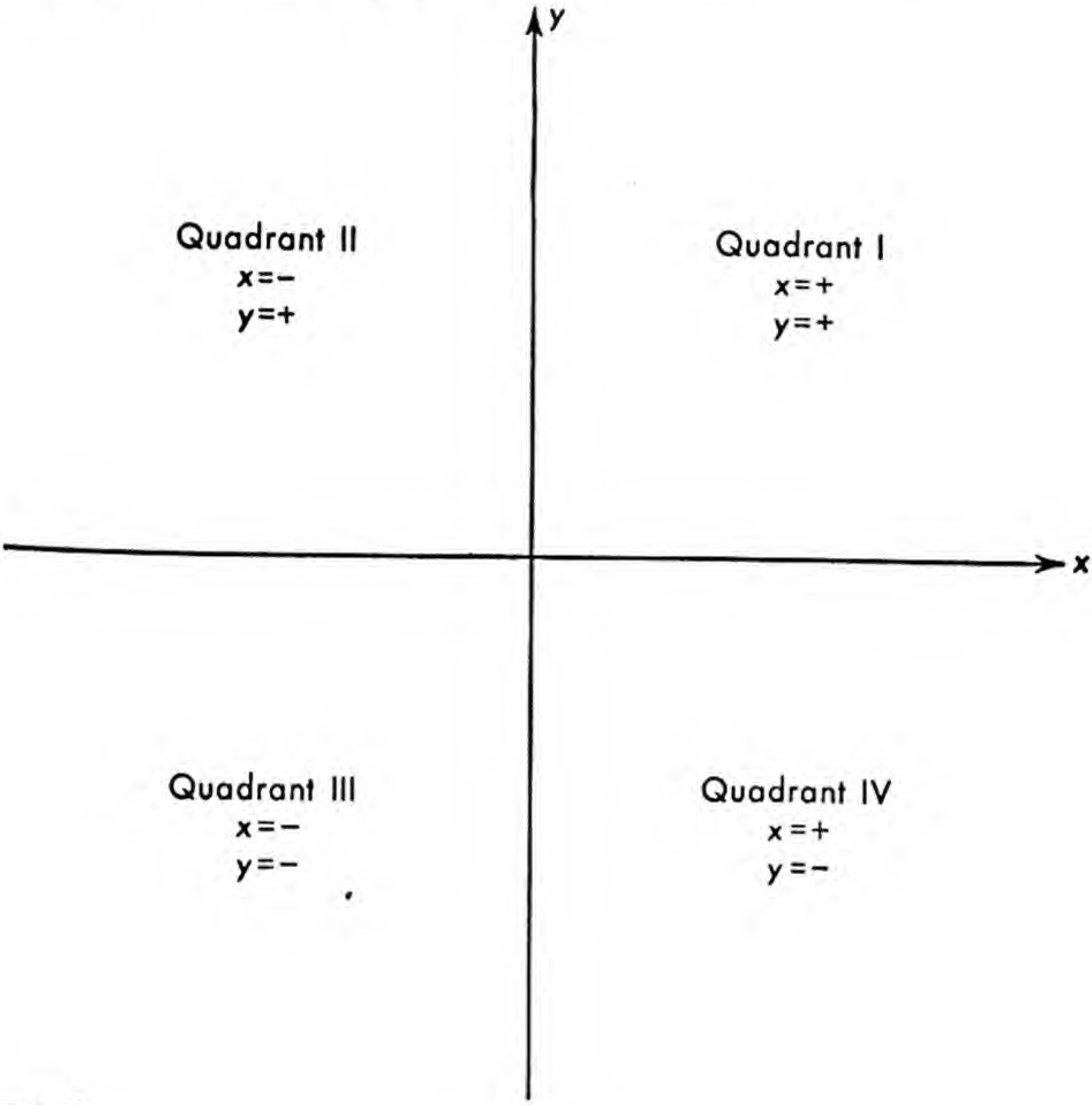


FIG. 4

Exercise 17

Use coordinate paper in working the following problems.

1. Plot the following points; place the coordinates of each point next to the point: (3, 2), (0, 4), (−1, 5), (−6, 0), (−2, −7), (4, −3), (0, 0).

2. Plot the following points; place the coordinates of each point next to the point: (2, 5), (−4, 3), (−4, −1), (0, −3), (1, −5), (6, 0).

3. Copy Fig. 5 and write the coordinates of each point to the right of the letter designating the point.

4. Draw the quadrilateral whose vertices are $(-3, 5)$, $(-6, -1)$, $(4, -7)$, $(0, 8)$.

5. Draw the trapezoid whose vertices are $(1, 7)$, $(-4, 5)$, $(-4, 0)$, $(1, -3)$.

6. What is the x of all points on the y-axis? What is the y of all points on the x-axis?

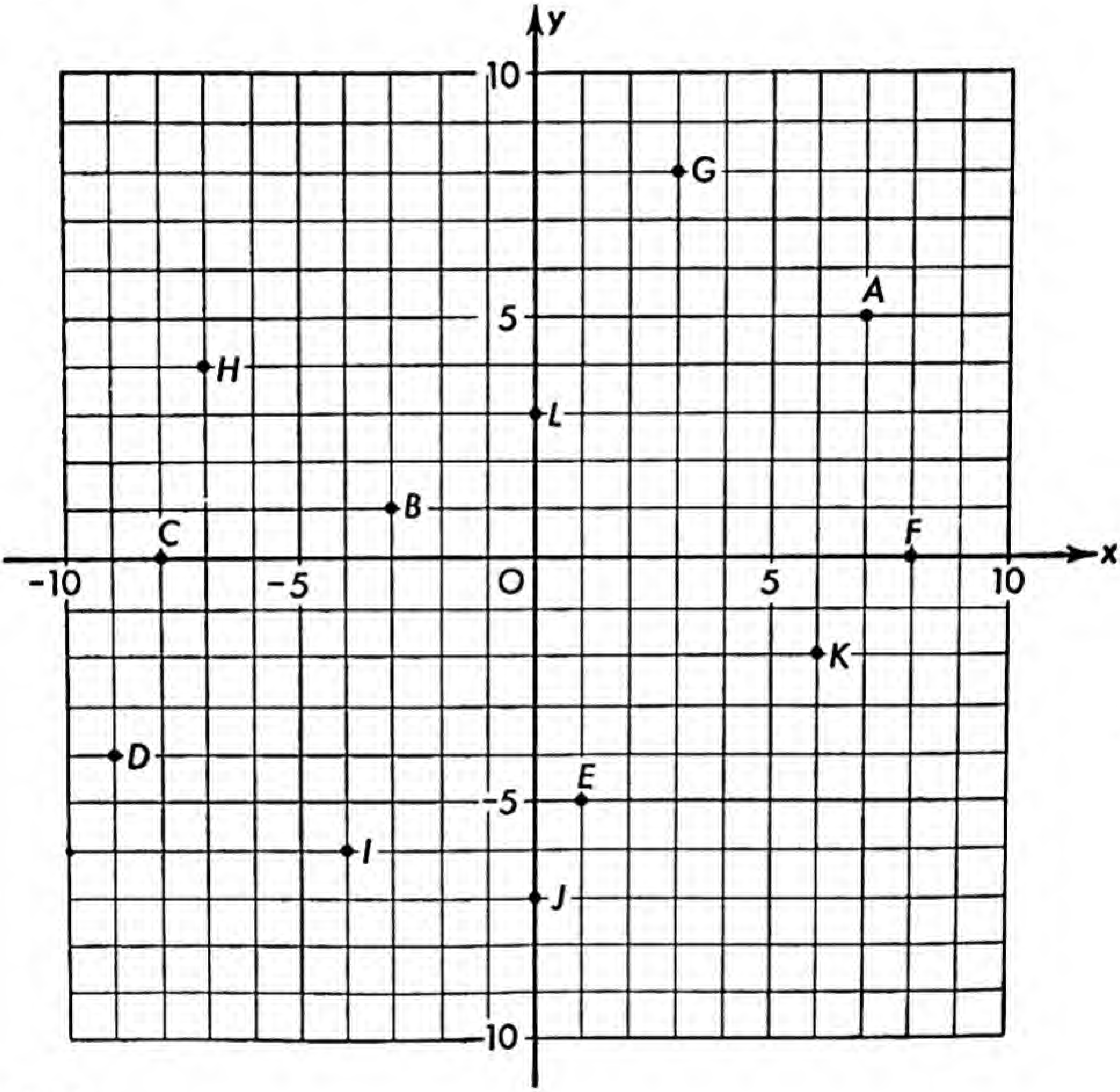


FIG. 5

7. Locate the points $(5, -4)$, $(5, -3)$, $(5, -1)$, $(5, 0)$, $(5, 2)$, $(5, 6)$. What can you conclude as to the position of all points having an x of 5?

8. Draw the line that passes through $(-3, 4)$ and is parallel to the x-axis. What is the value of the y of every point on this line?

9. In which quadrant does each of the following points lie if s is a negative number: (a) $(s, 4)$, (b) $(-1, s)$, (c) $(3, s)$, (d) $(-s, 5)$, (e) (s^2, s^3).

10. Consider the x-axis as an east-west line and let the y-axis represent a north-south line through St. Louis which is taken as the origin of the coordinate system. The general course of Highway 66 from St. Louis to Chicago may be indicated by the following pairs of coordinates of points on the road (the numbers represent miles): (0, 0), (27, 22), (27, 27), (33, 40), (33, 82), (48, 103), (79, 136), (111, 180), (111, 202), (140, 220). Plot the points and connect adjacent points with straight lines to indicate the general trend of the Illinois portion of Highway 66. (Choose a convenient scale such as 5 or 10 miles to one coordinate space.)

35. Graph of a function. Let y represent any function of x. To each value assigned to x, there will, in general, correspond a value of y. Each such pair of values can be used as the coordinates of a point in a plane. The graph of the function is a smooth curve passing through these points.

In selecting the values to be assigned to x, it is usually best to

1. Set $x = 0$.

2. Set $x = 1, 2, 3, \cdots$, continuing until the general trend of y is discovered.

3. Set $x = -1, -2, -3, \cdots$, until the behavior of y is determined.

4. When there is doubt as to the nature of the curve between two points, interpolate with fractional values of x, such as $\frac{1}{2}$, $-1\frac{1}{2}$, $2\frac{1}{4}$.

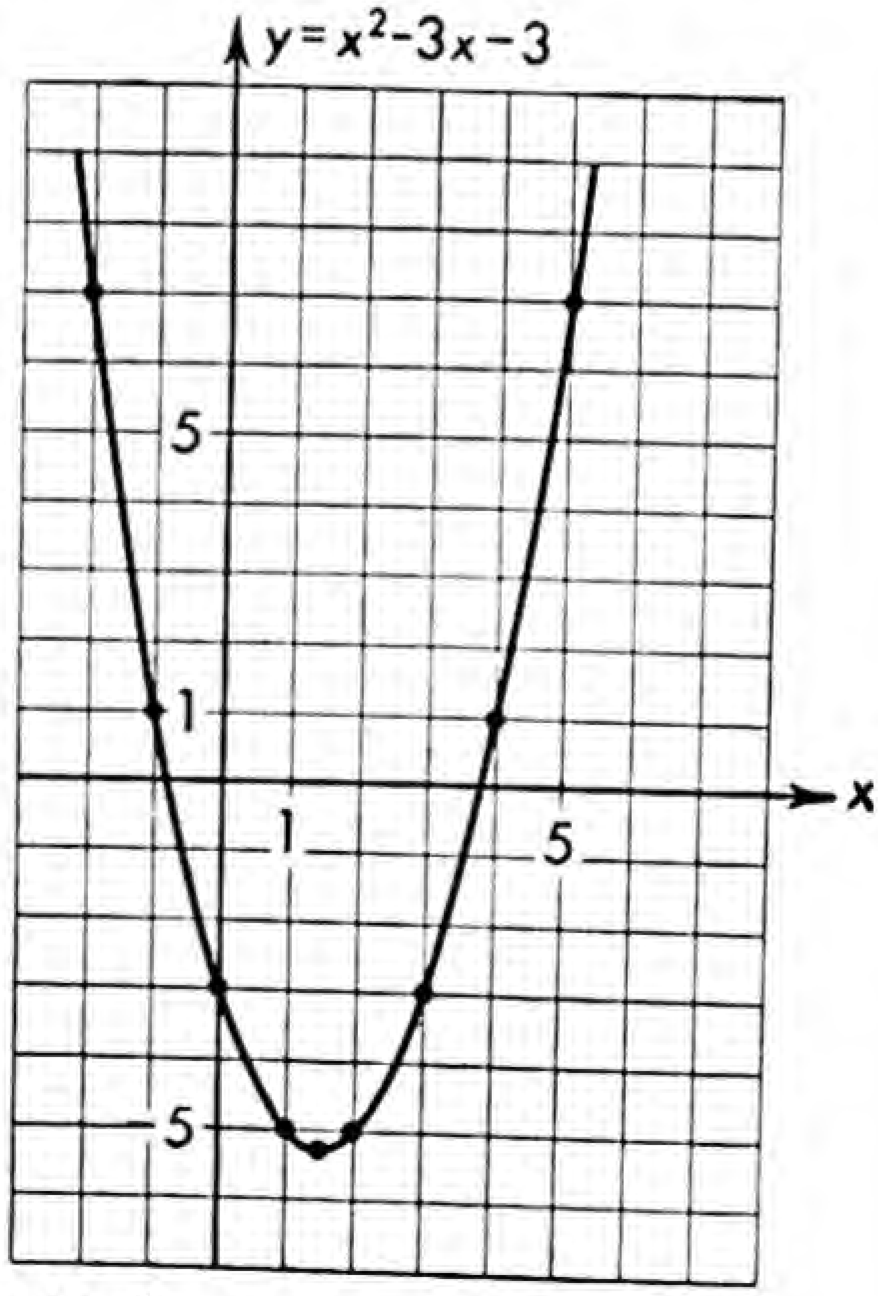


FIG. 6

Example 1. Graph the function of x: $x^2 - 3x - 3$.

Solution. Let y represent the function:

$$y = x^2 - 3x - 3.$$

If $x = 0$, then $y = 0^2 - 3(0) - 3 = -3$. If $x = 1$, then $y = 1^2 - 3(1) - 3 = -5$. By assigning several more values to x and computing the corresponding values of y, we form the following table.

x		-2	-1	0	1	2	3	4	5	$\frac{3}{2}$	
y		7	1	-3	-5	-5	3	1	7	$-\frac{21}{4}$	

Notice that, in order to determine more accurately the behavior of the curve between $x = 1$ and $x = 2$, we included the additional point $(1\frac{1}{2}, -5\frac{1}{4})$. The curve is called a **parabola.** From the graph we see that the function y is 0 (the curve crosses the x-axis) when x is $-.8$ and 3.8, approximately.

A **linear function** of x is an expression of the form $ax + b$, where a and b are constants and $a \neq 0$. It is proved in analytic geometry that the graph of every linear function of a single variable is a straight line (as implied by the word "linear"). When graphing a linear function, we need only two points to determine the line. It is wise, however, to plot a third point to check the accuracy of the other two.

Example 2. Graph the function of x: $\frac{7}{4}x + 1$.

Solution. Let $y = \frac{7}{4}x + 1$. If $x = 0$, $y = \frac{7}{4}(0) + 1 = 1$. Our graph will be more accurate if the second point is not too close to the first one. Setting $x = 4$, we find $y = 8$. As a check, set $x = -1$; then $y = -\frac{3}{4}$.

x		-1	0	4	
y		$-\frac{3}{4}$	1	8	

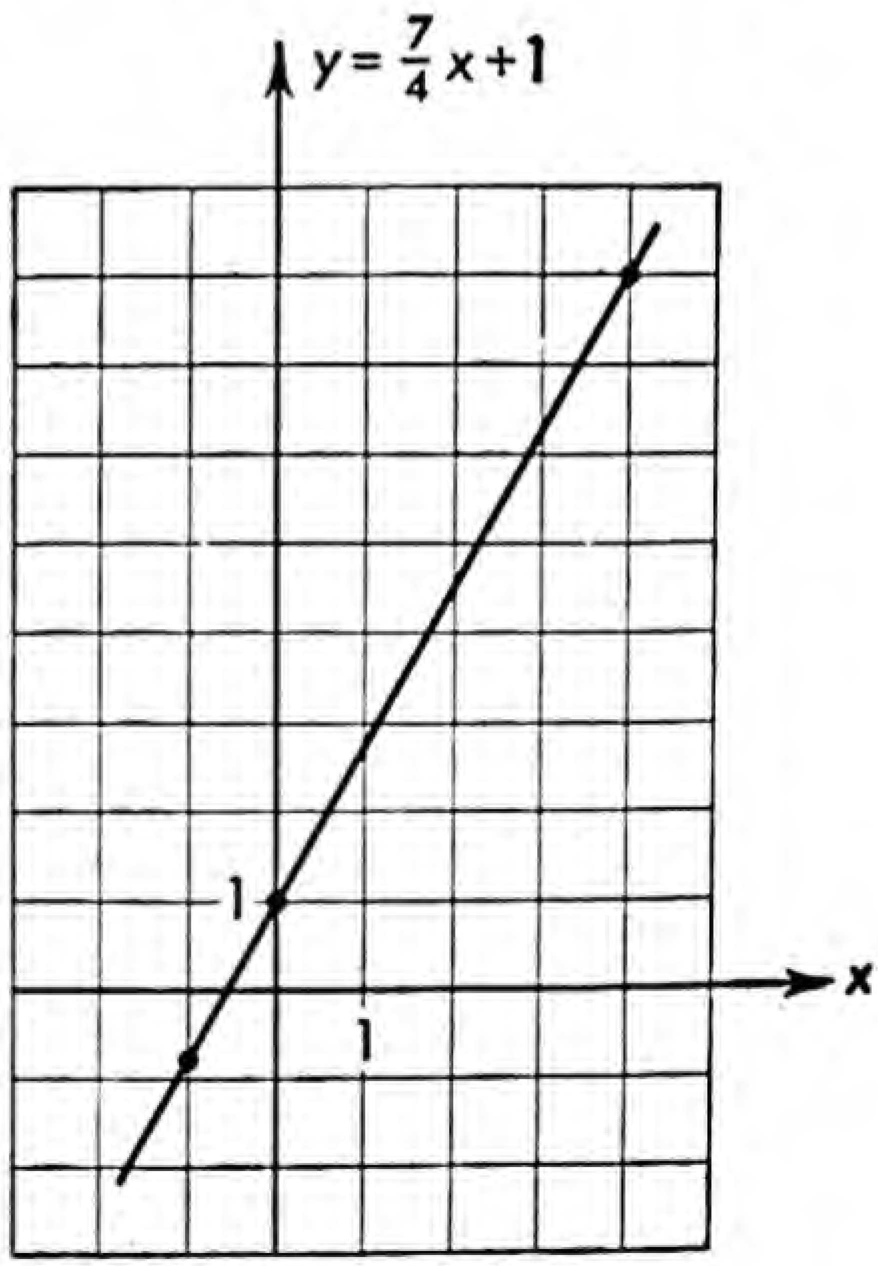


FIG. 7

36. Graph of a function not defined by a formula. Sometimes a function is defined by a table of corresponding values of the independent variable and the function. Such a table is usually found by experiment or mere observation. To graph this type of function, select a convenient unit for each axis, plot the points, and draw a

smooth* curve through them. It may be advisable to place the axes so that their intersection represents a pair of convenient values (not necessarily zero) of the two variables.

Exercise 18

Graph the following functions of x.

1. $3x - 5$. **2.** $x + 3$. **3.** $-2x - 1$. **4.** $-5x + 2$.
5. $-\frac{1}{2}x + 3$. **6.** $\frac{3}{2}x - 4$. **7.** $\frac{5}{3}x + 2$. **8.** $-\frac{3}{4}x - 1$.
9. $-2x$. **10.** x. **11.** $x^2 - 7$. **12.** $-x^2 + 5$.
13. $-x^2 + 4x$.

14. $x^2 - x - 7$. (Plot points for $x = -4, -3, -2, -1, 0, \frac{1}{2}, 1, 2, 3, 4, 5$.)

15. $x^2 + 2x - 5$. (Plot points for $x = -4, -3, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, 1, 2, 3$.)

16. x^3. (Plot points for $x = -2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2$.)

17. The amount of usable power developed by an ordinary dry-cell battery with given voltage depends upon the external resistance.

(*a*) Use the following data to graph the power as a function of the resistance.

Resistance (in ohms)	0	.01	.02	.03	.04	.05	.08	.12	.16	.20
Power (in watts)	0	6.3	9.2	10.5	11.1	11.25	10.7	9.3	8.2	7.2

(*b*) Read from the graph the power developed when the resistance is .10 ohm.

18. Under certain conditions, the range of a gun depends upon its elevation (the angle its muzzle makes with the horizontal).

(*a*) Use the following table to graph the range of a certain gun as a function of the elevation.

Elevation (in degrees)	5	10	20	25	30	35	40	45
Range (in miles)	1.7	3.4	6.4	7.7	8.7	9.4	9.8	10.0

(*b*) Use the graph to find the proper elevation for a range of 5 miles.

* A series of broken lines is preferable if it is obvious that the function changes abruptly.

19. The volume of a gas under various pressures is given by the following data.

Pressure (in pounds per square inch)	2	3	4	6	8	10	12
Volume (in cubic feet)	20	13.3	10	6.7	5	4	3.3

(*a*) Graph the volume as a function of the pressure.
(*b*) What pressure is needed to produce a volume of 9 cubic feet?

20. The following table gives the fuel economy of a certain automobile at various speeds.

Speed (in miles per hour)	0	5	10	15	20	30	40	50	60	70
Fuel economy (in miles per gallon)	0	12	17	19	20	19	17.7	16.1	14.2	12

Graph the fuel economy as a function of the speed.

21. The following table gives the number of radios in use in the United States at various times.

Year	1922	1923	1925	1930	1935	1940	1941	1942	1943
Radios (in millions)	.06	1.0	3.5	12	22	29	29.7	30.8	30.0

(*a*) Graph the number of radios as a function of time.
(*b*) Read from the graph the number of radios in use in 1932.

22. The following table gives the safe load for various lengths of a Douglas fir column 6 inches square.

Length (in feet)	6	8	10	12	14	16
Safe load (in tons)	13.2	12.2	10.9	9.4	7.6	5.6

(*a*) Graph the safe load as a function of the length.
(*b*) Read from the graph the safe load for a column of length 13 feet.

23. The following table gives the population of the United States at 10-year intervals beginning with the first census in 1790.

Census year	1790	1800	1810	1820	1830	1840	1850	1860
Population (in millions)	3.9	5.3	7.2	9.6	12.9	17.1	23.2	31.4

Census year	1870	1880	1890	1900	1910	1920	1930	1940
Population (in millions)	38.6	50.2	62.9	76.0	92.0	105.7	122.8	131.7

(*a*) Graph the population as a function of time.
(*b*) Use the graph to estimate the population in 1845.
(*c*) When was the population 100 millions?

24. The following table gives the Republican membership of the United States Senate following various biennial elections.

Election year

1920	1922	1924	1926	1928	1930	1932	1934	1936	1938	1940	1942	1944	1946
Number of Republicans													
59	51	54	48	56	48	36	25	17	23	28	38	38	51

Graph the number of Republicans as a function of time. The points should be connected by a series of straight lines rather than a smooth curve. Why?

37. Implicit functions. If x and y are the only two variables involved in an equation, then y is said to be an **implicit function** of x; also, x is an implicit function of y. For example, if

$$3x + y = 5, \tag{1}$$

then y is an implicit function of x (it is *implied* that for each value of x there is a corresponding value of y). Upon solving this equation for y in terms of x, we get

$$y = -3x + 5.$$

Now y is expressed explicitly in terms of x. When the equation involving x and y is written in the form $y = f(x)$, then we say that y is an **explicit function** of x.

Solving (1) for x in terms of y, we get

$$x = \frac{5 - y}{3}.$$

Now x is expressed as an explicit function of y. Sometimes we cannot express one variable as an explicit function of the other. For instance, if $y^7 + y^3 + y^2 + 7x = \pi x + 4$, then y cannot be expressed as an explicit function of x. Can x be expressed as an explicit function of y?

A solution of an equation *involving x and y is a pair of values of x and y that satisfies the equation.* We can find a solution of such an equation by assigning a value to *either* variable and then computing the value of the other variable.

Illustration 1. In the equation $2x + 3y = 8$, let us set $x = 6$. Then $12 + 3y = 8$, and $y = -\frac{4}{3}$. Hence $(x = 6, y = -\frac{4}{3})$ is *a solution* of the equation. If we set $y = 0$, then $x = 4$. Hence $(x = 4, y = 0)$ is another solution of the equation. How many solutions are there?

38. Graph of an equation. *The* **graph** *of an equation in two variables x and y is the curve * containing those points and only those points whose coordinates satisfy the equation.* Or, the graph is the totality of points whose coordinates (x, y) form solutions of the equation.

A linear equation in x and y is an equation of the form

$$Ax + By = C$$

where A, B, and C are constants. In analytic geometry it is proved that the graph of every linear equation in two variables is a straight line.

To graph a linear equation in x and y.

1. *Plot the point for which $x = 0$.*
2. *Plot the point for which $y = 0$.*
3. *Draw the line determined by these two points.*
4. *Check by finding a third solution and demonstrating that the corresponding point does lie on the line.*

* The word "curve" is usually used to include straight lines.

Example 1. Graph the equation $3x - 5y = 10$.

Solution. If $x = 0$, then $y = -2$. If $y = 0$, then $x = \frac{10}{3}$. Draw the line through these two points. To check, set $x = -2$. Then $y = -\frac{16}{5}$. (See Fig. 8.)

The student should realize that every solution of the given equation corresponds to some point on the line.

Example 2. Graph the equation $x = 4$.

Solution. Since y is not mentioned in the equation, its value can be anything. If $y = 0$, then $x = 4$. If $y = 3$, then $x = 4$. All points lie on a line parallel to and 4 units to the right of the y-axis.

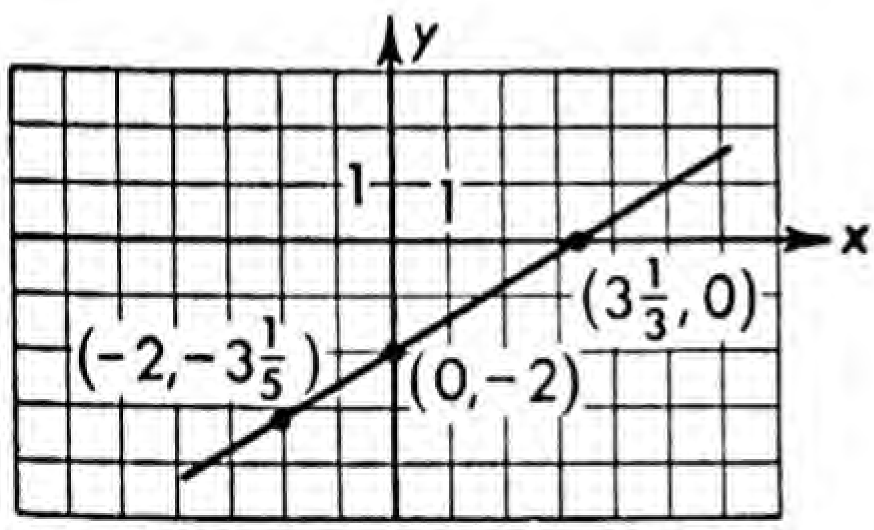


FIG. 8

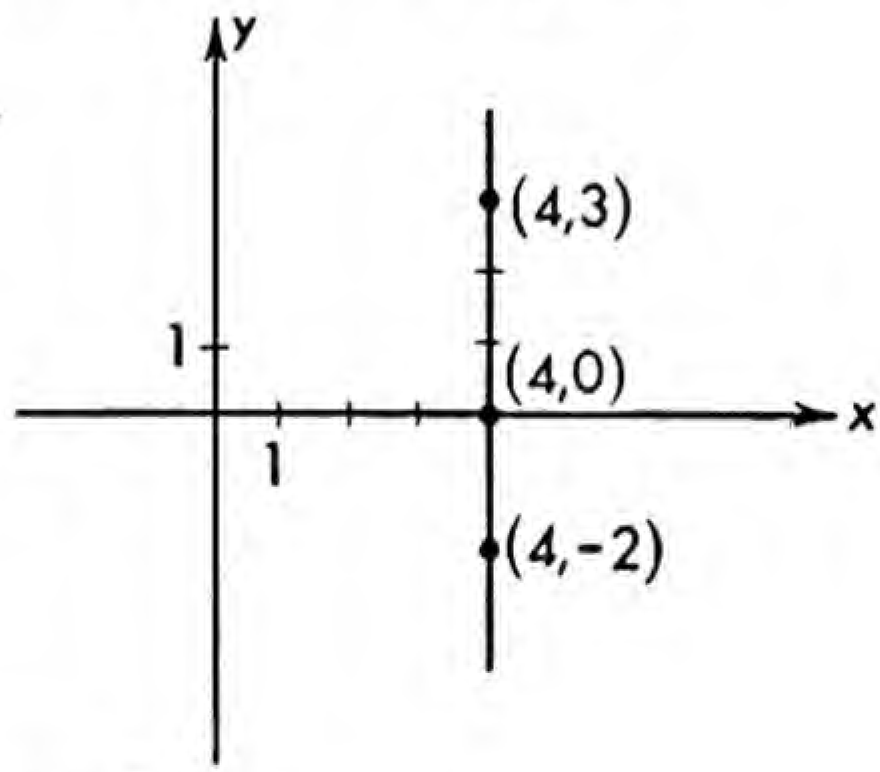


FIG. 9

Exercise 19

Graph the following equations.

1. $2x - 3y = 12$.
2. $4x + 5y = 20$.
3. $-3x - y = 6$.
4. $-x + 4y = 8$.
5. $7x + 2y - 8 = 0$.
6. $x + 3y + 7 = 0$.
7. $5x - 3y + 18 = 0$.
8. $3y - 4x + 15 = 0$.
9. $3x - y = 0$.
10. $x + y = 0$.
11. $4x + 3y = 0$.
12. $7x - 5y = 0$.
13. $y = 5$.
14. $x = -3$.
15. $2x + 7 = 0$.
16. $4y - 9 = 0$.
17. $x = 0$.
18. $y = 0$.
19. Given the equation $3x + 7y = 8$.
(*a*) Express y as an explicit function of x.
(*b*) Express x as an explicit function of y.
20. Given the equation $4x - 5y = 7$.
(*a*) Express y as an explicit function of x.
(*b*) Express x as an explicit function of y.

chapter 6 | *Systems of linear equations*

39. Graphic solution of a system of two linear equations in two unknowns. A solution of a system of equations is a set of values of the unknowns that simultaneously satisfies each equation of the system. For example, the system

$$\begin{cases} 2x + 3y = 11 \\ 7x - 5y = 23 \end{cases}$$

has the solution $(x = 4, y = 1)$.

To solve a system of equations means to find all its solutions or to show that it has no solution. The system

$$\begin{cases} x + y = 3 \\ x + y = 4 \end{cases}$$

has no solution because it is obviously impossible to find two numbers whose sum is 3 and also 4 at the same time.

To solve a system of two linear equations in x and y graphically.

1. *Graph the two equations on the same coordinate system.*
2. *Estimate the coordinates of the point of intersection of the graphs. These coordinates form the solution.*

When a system is solved graphically, the solution is apt to be only approximately correct because fractional values cannot be read accurately from a figure.

Example 1. Solve graphically: $\begin{cases} 12x + 5y = 25, & \textbf{(1)} \\ 7x - 10y = 28. & \textbf{(2)} \end{cases}$

Solution. The graphs of equations (1) and (2) are the lines l_1 and l_2 respectively. The two lines seem to cross at the point $(2.5, -1)$. Hence the graphic solution is $(x = 2.5, y = -1)$.

Comment. The exact solution (which may be found by the methods of Art. 40) is $(x = 2\frac{16}{31}, y = -1\frac{6}{155})$ which is approximately $(x = 2.516, y = -1.039)$. It is obviously impossible for anyone to read the exact solution from the graph. For our purposes in this chapter, a graphic solution that comes within 0.2 of the exact solution will be considered satisfactory.

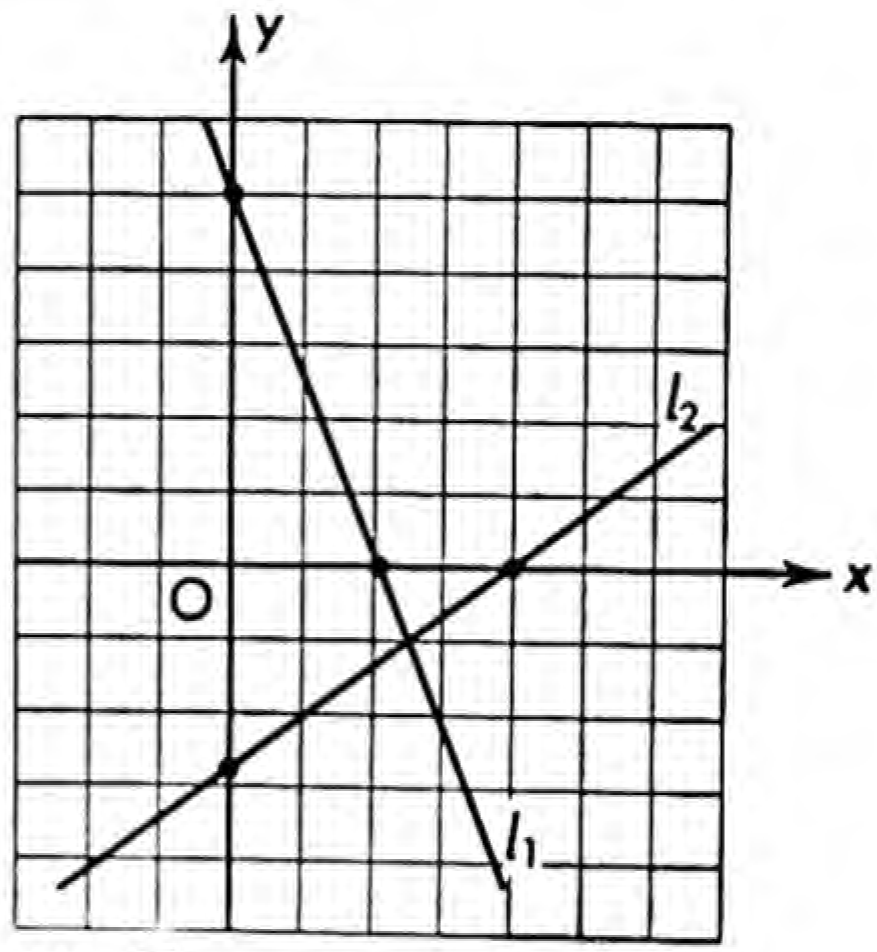


FIG. 10

The graphic solution of a pair of linear equations always leads to one of the three following situations.

I. *If the two lines* **intersect,** *the system has* **one solution** *and the equations are said to be* **consistent** *and independent.*

II. *If the two lines are* **parallel,** *the system has* **no solution** *and the equations are said to be* **inconsistent.**

III. *If the two lines* **coincide,** *the system has* **infinitely many solutions** *and the equations are said to be* **dependent.** Each solution of one equation is automatically a solution of the other, and therefore of the system.

Exercise 20

Solve graphically. State the number of solutions and identify each system as consistent and independent, inconsistent, or dependent. If there is only one solution, express each estimate to one decimal place. If there are infinitely many solutions, write any two of them.

1. $\begin{cases} x - y = -5, \\ 3x + 7y = 14. \end{cases}$

2. $\begin{cases} x + 3y = 6, \\ 2x + y = 8. \end{cases}$

3. $\begin{cases} 6x + 5y = 10, \\ 2x - 5y = 14. \end{cases}$

4. $\begin{cases} 9y + 4x = 24, \\ 6y - 5x = 30. \end{cases}$

5. $\begin{cases} 3x + 2y = -12, \\ x - y = 3. \end{cases}$

6. $\begin{cases} 5x - 6 = 0, \\ x + 4y + 4 = 0. \end{cases}$

7. $\begin{cases} 2y - 5 = 0, \\ y - 3x + 3 = 0. \end{cases}$

8. $\begin{cases} 2x + 3y + 6 = 0, \\ 4x - 5y + 10 = 0. \end{cases}$

9. $\begin{cases} x + 2y = 0, \\ 3x - 4y = 16. \end{cases}$

10. $\begin{cases} x + y = -2.6, \\ .7x - y = -4.2. \end{cases}$

11. $\begin{cases} x - y = 3, \\ 2x - 2y = 5. \end{cases}$

12. $\begin{cases} 3x = 4y + 9, \\ 8y = 6x - 18. \end{cases}$

13. $\begin{cases} 2x + 3y = 12, \\ 6x + 9y = 36. \end{cases}$

14. $\begin{cases} 2x - y = 4, \\ x = \frac{1}{2}y + 1. \end{cases}$

15. $\begin{cases} x + 3y = 6, \\ 3y = 3 - x. \end{cases}$

16. Graph the equation $3x + 2y = 12$. Multiply both sides by 2 and graph the new equation. Classify the two equations as consistent and independent, inconsistent, or dependent.

40. Algebraic solution of a system of two linear equations in two unknowns. The graphic method of solving a system of two equations in x and y usually gives results that are only approximately correct. If both equations are linear, an exact solution may be found by either of two algebraic methods, each of which involves a process called **elimination.** From the two given equations we derive several equivalent equations (Art. 28) the last of which involves only one of the unknowns. After finding the value of this unknown, we substitute it in one of the original equations to find the other unknown.

I. *Elimination by substitution.* Solve one equation for one unknown in terms of the other and substitute this result in the other equation.

Example 1. Solve by elimination by substitution:

$$\begin{cases} 4x + 2y = -1, & (1) \\ 5x - 3y = 7. & (2) \end{cases}$$

Solution. We shall eliminate y. Solve (1) for y in terms of x:

$$y = \frac{-4x - 1}{2}. \qquad (3)$$

Substitute this expression for y in (2):

$$5x - 3\left(\frac{-4x - 1}{2}\right) = 7.$$

Multiply through by 2 to clear of fractions:

$$10x + 12x + 3 = 14,$$
$$22x = 11,$$
$$x = \tfrac{1}{2}.$$

To find y, substitute $x = \frac{1}{2}$ in (3):

$$y = \frac{-4(\frac{1}{2}) - 1}{2} = \frac{-2 - 1}{2} = -\frac{3}{2}.$$

The solution of the system is $(x = \frac{1}{2}, y = -\frac{3}{2})$.

Check. Substituting these values for x and y in the *original* equations we get

$$4(\tfrac{1}{2}) + 2(-\tfrac{3}{2}) = -1,$$
$$2 - 3 = -1, \quad \text{True.}$$

and

$$5(\tfrac{1}{2}) - 3(-\tfrac{3}{2}) = 7,$$
$$\tfrac{5}{2} + \tfrac{9}{2} = 7. \quad \text{True.}$$

Comment. In choosing the unknown that is to be eliminated by substitution, avoid fractions whenever possible. For example, the equation $x + 3y = 7$ should be solved for x in terms of y rather than for y in terms of x.

II. *Elimination by addition or subtraction.* Multiply each equation by a properly chosen constant so that one unknown will be eliminated when these two derived equations are added or subtracted.

Example 2. Solve by elimination by addition or subtraction:

$$\begin{cases} 4x + 2y = -1, & (1) \\ 5x - 3y = 7. & (2) \end{cases}$$

Solution. We shall eliminate x.

Multiply (1) by 5:	$20x + 10y = -5.$	(3)
Multiply (2) by 4:	$20x - 12y = 28.$	(4)
Subtract, (3) − (4):	$22y = -33.$	(5)
Divide (5) by 22:	$y = -\frac{33}{22} = -\frac{3}{2}.$	

To find x, substitute $y = -\frac{3}{2}$ in either of the original equations. If we use (1), we get $4x + 2(-\frac{3}{2}) = -1$, $4x - 3 = -1$, $4x = 2$, $x = \frac{1}{2}$.

The solution ($x = \frac{1}{2}$, $y = -\frac{3}{2}$) should be checked in both original equations.

Instead of eliminating x from the two given equations, we could have eliminated y by multiplying (1) by 3, multiplying (2) by 2, and then *adding* the two derived equations.

If, in our attempt to eliminate one unknown, the other unknown is also eliminated, then the two equations are either inconsistent or dependent. *If the elimination results in a contradiction such as* $0 = 5$, *the equations are inconsistent. If the elimination results in the identity* $0 = 0$, *the equations are dependent.* These facts will be discussed more in detail in Art. 182.

Frequently we encounter equations of the type $\frac{a}{x} + \frac{b}{y} = c$. Although this equation is not linear in x and y,* the method of elimination by addition or subtraction can be used in solving a system of two equations of this kind. It is advisable to avoid clearing of fractions until after the elimination has been performed.

Example 3. Solve for x and y:

$$\begin{cases} \dfrac{1}{x} + \dfrac{3}{y} = 8, & (1) \\ \dfrac{5}{x} - \dfrac{1}{y} = 4. & (2) \end{cases}$$

Solution. We shall eliminate y.

Multiply (2) by 3: $$\frac{15}{x} - \frac{3}{y} = 12. \qquad (3)$$

Add (1) and (3):

$$\frac{16}{x} = 20,\ 16 = 20x,\ x = \frac{16}{20} = \frac{4}{5}.$$

Substitute $\frac{4}{5}$ for x in (1):

$$\frac{1}{\frac{4}{5}} + \frac{3}{y} = 8,\quad \frac{5}{4} + \frac{3}{y} = 8,\quad \frac{3}{y} = 8 - \frac{5}{4} = \frac{27}{4},\quad \frac{y}{3} = \frac{4}{27},\quad y = \frac{4}{9}.$$

The student should check the solution ($x = \frac{4}{5}$, $y = \frac{4}{9}$) in both of the original equations.

* It is linear in $\frac{1}{x}$ and $\frac{1}{y}$.

Exercise 21

Solve by elimination by substitution and check.

1. $\begin{cases} x + 3y = 7, \\ 2x + 5y = 11. \end{cases}$
2. $\begin{cases} 4x + y = 10, \\ 7x + 3y = 25. \end{cases}$
3. $\begin{cases} 2x - 3y = 4, \\ 5x - 6y = 7. \end{cases}$
4. $\begin{cases} 7x - 5y = 1, \\ 3x + 3y = 1. \end{cases}$
5. $\begin{cases} 6x + 2y = 15, \\ 4x - 7y = 10. \end{cases}$
6. $\begin{cases} 2x - 6y = 1, \\ 3x - 5y = 2. \end{cases}$

Solve by elimination by addition or subtraction and check.

7. $\begin{cases} x + y = 10, \\ x - y = 8. \end{cases}$
8. $\begin{cases} x - 3y = 7, \\ x - 5y = 9. \end{cases}$
9. $\begin{cases} 2x - 7y = 4, \\ 6x - 5y = 0. \end{cases}$
10. $\begin{cases} 3x + 4y = 0, \\ 5x + 6y = 0. \end{cases}$
11. $\begin{cases} 5y + 2x = 1, \\ 4y + 3x = 2. \end{cases}$
12. $\begin{cases} 2y + 4x - 5 = 0, \\ 7y - 6x + 5 = 0. \end{cases}$

Solve algebraically by either method and check. If the equations are inconsistent or dependent, check by graphing.

13. $\begin{cases} x + 4y = 1, \\ 3x - 2y = 24. \end{cases}$
14. $\begin{cases} 3x + y = 1, \\ 2x + 4y = 5. \end{cases}$
15. $\begin{cases} 6x + 7y = 3, \\ 8x + 9y = 1. \end{cases}$
16. $\begin{cases} 5x - 3y = 11, \\ 7x + 6y = 12. \end{cases}$
17. $\begin{cases} 5x = 6y + 4, \\ 9y = 2x - 6. \end{cases}$
18. $\begin{cases} 2y = 8x - 3, \\ 3y = 5x - 1. \end{cases}$
19. $\begin{cases} .3x + .1y = .9, \\ .5x - .2y = .4. \end{cases}$
20. $\begin{cases} .1x + .5y = 1.4, \\ .3x + .7y = 2.6. \end{cases}$
21. $\begin{cases} \dfrac{x + y + 6}{2x + y + 9} = \dfrac{2}{3}, \\ \dfrac{x + 4y + 1}{5x + 3y + 1} = 1. \end{cases}$
22. $\begin{cases} \dfrac{x + y}{4x + 5y + 1} = \dfrac{1}{5}, \\ \dfrac{x + 3}{x + 5} = \dfrac{y + 1}{y + 2}. \end{cases}$
23. $\begin{cases} \dfrac{x}{5} - \dfrac{2y}{3} = \dfrac{1}{2}, \\ \dfrac{x}{4} - \dfrac{y}{6} = \dfrac{3}{8}. \end{cases}$
24. $\begin{cases} x + \dfrac{y}{4} = \dfrac{3}{8}, \\ \dfrac{5x}{12} + \dfrac{y}{6} = \dfrac{1}{4}. \end{cases}$
25. $\begin{cases} 20y + 12x = 60, \\ 9x + 15y = 45. \end{cases}$
26. $\begin{cases} 3x + 6y = 10, \\ 5x + 10y = 20. \end{cases}$
27. $\begin{cases} 3x - 6y = 7, \\ 4x - 8y = 9. \end{cases}$
28. $\begin{cases} 7x = 2y + 10, \\ 6y = 21x - 30. \end{cases}$

29. $\begin{cases} 8x + 20y = 25, \\ 2x + 5y = 12. \end{cases}$

30. $\begin{cases} 4x - 8 = 3y, \\ 6y + 16 = 8x. \end{cases}$

31. $\begin{cases} \dfrac{6}{x} + \dfrac{1}{y} = 7, \\ \dfrac{6}{x} - \dfrac{1}{y} = 1. \end{cases}$

32. $\begin{cases} \dfrac{2}{x} + \dfrac{3}{y} = 2, \\ \dfrac{5}{x} + \dfrac{7}{y} = 3. \end{cases}$

33. $\begin{cases} \dfrac{1}{x} - \dfrac{2}{y} = 2, \\ \dfrac{3}{x} - \dfrac{5}{y} = 8. \end{cases}$

34. $\begin{cases} \dfrac{4}{x} + \dfrac{1}{y} = 11, \\ \dfrac{3}{x} + \dfrac{1}{2y} = 5. \end{cases}$

Solve for x and y.

35. $\begin{cases} ax + by = a^3, \\ bx + ay = b^3. \end{cases}$

36. $\begin{cases} 2ax + 5y = 17a + 5, \\ ax + 3y = 9a + 3. \end{cases}$

37. $\begin{cases} ax - by - a^2 + b^2 = 0, \\ bx + ay - 2ab = 0. \end{cases}$

38. $\begin{cases} cx + ad = a(y + b), \\ d(cx + ab) = ab(y + d). \end{cases}$

41. Three linear equations in three unknowns. A system of three linear equations in three unknowns may be consistent and independent, inconsistent, or dependent. In this chapter we shall consider only the consistent and independent case, which has but one solution.

Example 1. Solve for x, y, z:

$$\begin{cases} 4x - y + 5z = 10, & (1) \\ 6x + y - 4z = -4, & (2) \\ 4x - 2y + 9z = 17. & (3) \end{cases}$$

Solution. We shall eliminate y from two pairs of equations.

Add (1) and (2): $10x + z = 6.$ (4)

Double (2) and add (3): $16x + z = 9.$ (5)

Subtract, (4) − (5): $-6x = -3$, $x = \frac{1}{2}$.

Substitute $x = \frac{1}{2}$ in (4): $z = 1$.

Substitute $x = \frac{1}{2}$, $z = 1$ in (1): $2 - y + 5 = 10$, $y = -3$.

The solution is ($x = \frac{1}{2}$, $y = -3$, $z = 1$), which should be checked in all three original equations.

To solve three linear equations in three unknowns.

1. *Decide by inspection which unknown can be most easily eliminated.*

2. *Eliminate this unknown from one pair of the equations; eliminate the same unknown from another pair of the original equations.*

3. *Solve the two derived equations for the two remaining unknowns.*

4. *Substitute the values of the two unknowns found in Step 3 in the simplest of the original equations and solve for the third unknown.*

Comment. If one of the unknowns does not appear in one of the original equations, then eliminate this unknown from the other two equations. This reduces the original system of three equations to a new system of two equations in only two unknowns.

To solve a system of four linear equations in four unknowns, use a method analogous to that for three equations in three unknowns.

Exercise 22

Solve and check.

1. $\begin{cases} 2x - 3y + z = 0, \\ 3x + 4y - z = 17, \\ x + y = 5. \end{cases}$

2. $\begin{cases} x + y + z = 4, \\ 2x - z = 0, \\ 4x - y + 2z = 7. \end{cases}$

3. $\begin{cases} x + z = 2, \\ 2x + y - 4z = 0, \\ 3x + y + 5z = 14. \end{cases}$

4. $\begin{cases} 3x + 4y + z = 4, \\ 5x - 2y - z = 8, \\ 7x + 3y + z = 10. \end{cases}$

5. $\begin{cases} x + 5y + 6z = 11, \\ x + 4y + 8z = 11, \\ x + 3y + 4z = 8. \end{cases}$

6. $\begin{cases} 3x + 2y + 4z = 4, \\ x - 3y - 2z = 1, \\ 2x - 5y - 3z = 1. \end{cases}$

7. $\begin{cases} 3x + 2y - z = 2, \\ 4x + 7y + 2z = 12, \\ 5x - 3y + z = 1. \end{cases}$

8. $\begin{cases} 4x - 2y - 3z = 12, \\ 5x + 2y + 2z = 0, \\ 7x + 2y + 4z = -4. \end{cases}$

9. $\begin{cases} 5x + 2y - 3z = 0, \\ 6x - 3y + 8z = -4, \\ 7x + y - 5z = 1. \end{cases}$

10. $\begin{cases} 6x + 7y + 9z = 6, \\ x + 2y + 3z = 2, \\ 4x + 3y + 5z = 0. \end{cases}$

11. $\begin{cases} 2x + 3y + z = 4, \\ 3x + 4y + 2z = 6, \\ 2x - 5y - 3z = 4. \end{cases}$

12. $\begin{cases} 7r + s + 3t = 23, \\ 5r + 2s + 6t = 28, \\ 3r + 8s + 7t = 10. \end{cases}$

13. $\begin{cases} x + y + z = 6, \\ x + 4y = 3, \\ 2x + 5y = 4. \end{cases}$

14. $\begin{cases} 2x - 7y + 5z = 3, \\ 3x + 2y - 2z = 4, \\ 7x - 3y - 3z = -7. \end{cases}$

15. $\begin{cases} a - 2b - 3c - d = 5, \\ 3a + b + 4c + d = 4, \\ 2a - 4b - c = 3, \\ a + 5b = 2. \end{cases}$

16. $\begin{cases} a + b + c + d = 2, \\ 2a - 3b + 4c + d = 3, \\ 3a - 2b - 3c + d = 4, \\ 2a + 4b - 2c + 3d = 5. \end{cases}$

42. Stated problems. Our principal justification for learning how to solve a system of equations lies in the fact that many stated problems involve several unknowns and several conditions. When these conditions are translated into algebraic language, we may have two or more equations which must be satisfied simultaneously by a single set of values of the unknowns.

If several unknowns are mentioned, represent each one of them by a different letter. Every verbal condition states or implies that two quantities are equal. Express this equality by means of an equation. After all conditions have been translated into equations, solve the system and check the results.

The student should review Art. 30 before proceeding.

Example 1. Working together, A and B can paint a house in 30 hours. After they work together for 18 hours, A becomes ill and B requires 22 additional hours to finish the task alone. How long would it take for each one to do the job alone?

Solution. Let x represent the number of hours it takes A to paint the house alone and let y equal the number of hours required for B to do the job unassisted.

Then $\frac{1}{x}$ is the fractional part of the task performed by A in 1 hour, and $\frac{1}{y}$ is the part of the job accomplished by B in 1 hour. The part of the task done by A in 30 hours is $30\left(\frac{1}{x}\right)$ or $\frac{30}{x}$. The job is finished if the sum of the fractional parts performed by the workers is equal to 1.

Hence,
$$\frac{30}{x} + \frac{30}{y} = 1 \tag{1}$$
and
$$\frac{18}{x} + \frac{18}{y} + \frac{22}{y} = 1. \tag{2}$$

Solving this system by the method of page 70 gives us ($x = 66$, $y = 55$).

Check. In (1),
$$\tfrac{30}{66} + \tfrac{30}{55} = 1.$$
$$\tfrac{5}{11} + \tfrac{6}{11} = 1. \quad \text{True.}$$

In (2),
$$\tfrac{18}{66} + \tfrac{40}{55} = 1.$$
$$\tfrac{3}{11} + \tfrac{8}{11} = 1. \quad \text{True.}$$

Example 2. In a certain "amateur" football game, a team executed 17 scoring plays for a total of 60 points. The total number of conversions and safeties exceeded the number of touchdowns by 1. No field goals were made. How was the scoring accomplished?

Solution. Let $t =$ number of touchdowns (each worth 6 points),
$c =$ number of conversions (each worth 1 point),
$s =$ number of safeties (each worth 2 points).

Then,
$$\begin{cases} t + c + s = 17, \\ 6t + c + 2s = 60, \\ c + s = t + 1. \end{cases}$$

Solving, we get ($t = 8$, $c = 6$, $s = 3$), which should be checked by the student.

Exercise 23

Solve and check.

1. A grocer sold 3 quarts of milk and 2 pounds of butter to a customer for \$1.90. Another customer paid \$1.40 for 4 quarts of milk and 1 pound of butter. Find the price of 1 quart of milk and also of 1 pound of butter.

2. In playing a basketball game, a certain team scored 27 times for a total of 48 points. How many free throws were scored? How many field goals? (Each free throw counts 1 point. Each field goal counts 2 points.)

3. A man presented to a bank cashier a check for x dollars and y cents. The cashier made an error and gave the man y silver dollars and x pennies, a total of 10 coins. Find the amount of the check if the cashier shortchanged himself to the extent of \$1.98.

4. In a certain professional football game a team executed 12 scoring plays for a total of 43 points. The number of touchdowns and field goals was double the number of conversions. Find the number of touchdowns (6 points each), the number of conversions (1 point each), and the number of field goals (3 points each). No safeties were scored.

5. Flying with the wind, an airplane traveled 200 miles in 2 hours. The return trip, against the wind, required $2\frac{1}{2}$ hours. Find the speed of the airplane in still air and the speed of the wind.

6. A man rows 1 mile downstream on a river in $\frac{1}{3}$ hour. He requires another $\frac{1}{2}$ hour to row back upstream to his starting point. Find the speed of the current and the rate the man can row in still water.

7. A Democrat says to a Republican, "Give me one dollar and I'll have as much as you have." Says the Republican, "No, you give me a dollar and I'll have twice as much as you." How much did each man have?

8. If 8 furlongs plus 2 leagues equals 7 miles and if 2 furlongs plus 1 league equals $3\frac{1}{4}$ miles, find the number of miles in 1 furlong. In 1 league.

9. A man weighing 180 pounds sitting on one end of a teeter board 6 feet from the fulcrum balances a woman and a small boy on the other side at distances of 8 feet and 4 feet respectively from the fulcrum. If the woman and the boy change places, the man must move 2 feet closer to the fulcrum to preserve the balance. Find the weight of the woman and that of the boy.
Hint. See suggestion for problem 27, page 50.

10. If 1 is added to both numerator and denominator of a certain fraction, its value becomes $\frac{3}{5}$. If 1 is subtracted from numerator and denominator of the fraction, its value becomes $\frac{1}{2}$. Find the fraction.

11. If machines A and B work together, they can perform a certain task in 20 hours. If A works for 5 hours and B for 14 hours, the task would be half finished. How long would it take each machine alone to perform the task?

12. A mixture of 3 parts peanuts and 1 part cashews sells for 44 cents per pound. A mixture of 5 parts peanuts and 1 part cashews sells for 40 cents per pound. Find the price of a pound of peanuts. Of cashews.

13. Two track men run at constant speeds around a circular 660-foot track. When they run in opposite directions, they meet every 22 seconds. When they run in the same direction, the faster runner passes the slower one every 110 seconds. Find the speed of each runner.

14. The equation $y = mx + b$ is satisfied by $(x = 2, y = 3)$ and $(x = 5, y = 9)$. Find the values of m and b.
Hint. Substitute the sets of values in the equation and obtain $3 = 2m + b$ and $9 = 5m + b$. Solve this system.

15. A man invests \$10,000 in stocks, bonds, and a savings account which draw interest rates of 6%, 3%, and 2%, respectively. His annual return from the stocks and savings account is equal to the interest on the bonds. Find the three sums invested if the yearly interest is \$360.

16. Working together, A, B, and C can perform a task in 2 days. If A and B work together, they can do the job in 4 days. If B and C work together, they can perform the task in 3 days. How long would it take each man to do the job alone?

17. In making a vacation trip, a man traveled by plane at 150 miles per hour, by train at 50 miles per hour, and by auto at 30 miles per hour. The entire trip covered 758 miles and required 7 hours. The traveler spent 2 hours more in the plane than on the train. Find the time spent in each vehicle of transportation.

18. If the larger of two numbers is divided by the smaller, the quotient is 5 and the remainder is 4. If the smaller is divided by the larger, the ratio is $\frac{3}{16}$. Find the numbers.
Hint. Recall formula (1), Art. 11.

19. The sum of the digits of a three-digit number is 14. If the digits are reversed, the new number is 594 less than the original one. The hundreds' digit equals the sum of the tens' and units' digits. Find the number.
Hint. If h is the hundreds' digit, t the tens' digit, and u the units' digit, then the number is $(100h + 10t + u)$. For example, $234 = 2(100) + 3(10) + 4$.

20. A three-digit number is equal to 19 times the sum of its digits. If the digits are reversed, the new number exceeds the original one by 297. The tens' digit is one more than the sum of the hundreds' and units' digits. Find the number.

43. Solution of two linear equations by determinants. Every system of two linear equations in x and y can be written in the following general form.

$$\begin{cases} ax + by = E, & (1) \\ cx + dy = F, & (2) \end{cases}$$

where a, b, c, d, E, F are any numbers.

We shall eliminate y by subtraction.

$$\begin{array}{ll} \text{Multiply (1) by } d\text{:} & adx + bdy = Ed. \\ \text{Multiply (2) by } b\text{:} & bcx + bdy = Fb. \\ \text{Subtract:} & adx - bcx = Ed - Fb. \\ & x(ad - bc) = Ed - Fb. \end{array}$$

$$x = \frac{Ed - Fb}{ad - bc}, \tag{3}$$

provided $(ad - bc)$ is not zero.

By a similar process, we obtain

$$y = \frac{aF - cE}{ad - bc}. \tag{4}$$

Equations (3) and (4) represent the solution of the general system [(1), (2)]. These results can be most easily remembered by means of the following useful device. The symbol $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called a **determinant.** The numbers a, b, c, d are the **elements** of the determinant. The determinant is said to be of the **second order** because it contains two (horizontal) rows and two (vertical) columns. The value or expansion of this symbol is defined to be $(ad - bc)$, i.e.,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The elements a and d form the principal diagonal of the determinant.

We can say that the value of a second-order determinant is equal to the product of the elements in its principal diagonal minus the product of the elements in its other (secondary) diagonal.

Illustration 1. $\begin{vmatrix} 5 & -6 \\ 3 & 7 \end{vmatrix} = 5 \cdot 7 - 3 \cdot (-6) = 35 + 18 = 53.$

Illustration 2. $\begin{vmatrix} 4 & 0 \\ 9 & -2 \end{vmatrix} = 4 \cdot (-2) - 9 \cdot 0 = -8.$

Having defined the value of a determinant, we now see that equations (3) and (4), which represent the solution of the system [(1), (2)], can be written in the form

$$x = \frac{\begin{vmatrix} E & b \\ F & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & E \\ c & F \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}},$$

provided the denominator determinant, which is called the **determinant of the coefficients,** is not zero. If this determinant is zero, the system is either inconsistent or dependent. These exceptional cases will be discussed more fully in Art. 182.

Our results can be stated in the following rule.

Cramer's rule. *Each unknown is equal to the quotient of two determinants:*

1. *The denominator* (in each case) *is the determinant of the coefficients.*

2. *The numerator, for any unknown, is obtained from the denominator by substituting the constant terms for the coefficients of this unknown.*

Comment. It is to be understood that before applying Cramer's rule, we must write the equations in a symmetric arrangement like the system [(1), (2)], with *the constant terms alone on the right side.*

Example 1. Solve by determinants: $\begin{cases} 2x - 5y + 15 = 0, \\ 4y - 3x + 16 = 0. \end{cases}$

Solution. Rewrite the equations in standard form with the constant terms on the right side:

$$\begin{cases} 2x - 5y = -15, \\ 3x - 4y = 16. \end{cases}$$

Apply Cramer's rule:

$$x = \frac{\begin{vmatrix} -15 & -5 \\ 16 & -4 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & -4 \end{vmatrix}} = \frac{60 + 80}{-8 + 15} = \frac{140}{7} = 20.$$

$$y = \frac{\begin{vmatrix} 2 & -15 \\ 3 & 16 \end{vmatrix}}{\begin{vmatrix} 2 & -5 \\ 3 & -4 \end{vmatrix}} = \frac{32 + 45}{7} = \frac{77}{7} = 11.$$

Exercise 24

Solve for x and y by using determinants.

1. $\begin{cases} 3x + 4y = 7, \\ 5x + 6y = 8. \end{cases}$
2. $\begin{cases} 8x - 7y = 30, \\ 5x - 4y = 21. \end{cases}$
3. $\begin{cases} 4x - 3y = 10, \\ 6x + 2y = 15. \end{cases}$
4. $\begin{cases} 7x + 3y + 2 = 0, \\ 3x - 7y + 5 = 0. \end{cases}$
5. $\begin{cases} 7x + 8y = 0, \\ 3y + 2x = -10. \end{cases}$
6. $\begin{cases} 6x - 8y = -1, \\ 3x - 6y = 5. \end{cases}$

Problems 7 to 20. Solve problems 7 to 20 in Exercise 21 by using determinants.

44. Third-order determinants.

The symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

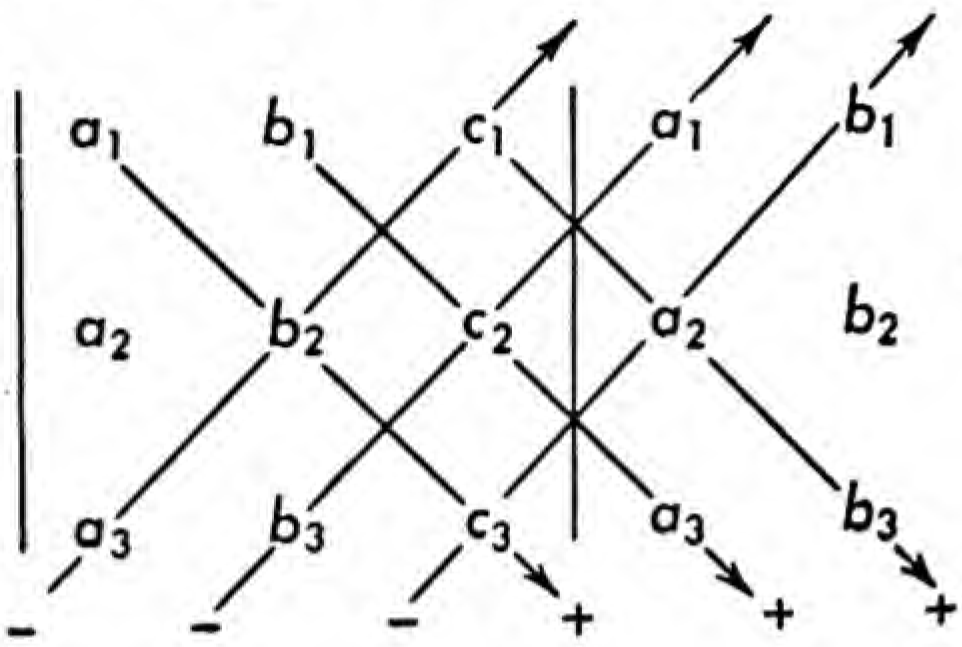
is a determinant of the third order and is defined to be equal to

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1.*$$

Notice that each of the six terms is the product of three elements, one from each row and, at the same time, one from each column.

* The symbol a_1 means "a with the subscript one." It is usually read "a sub one," or merely "a one." The symbols a_1 and a_2 can represent any numbers, such as 4 and -7. Subscripts are frequently used to simplify the notation. In this case each a represents an element in the first column while its subscript indicates the row.

The expansion may be remembered by use of the adjoining diagram. Rewrite the first two columns to the right of the determinant. Multiply as indicated by the arrows and (1) write the product of the three elements with no change of sign if the arrow points downward, and (2) write the product of the three elements with a change of sign if the arrow points upward.



This method of expansion applies to third-order determinants but *not to those of higher order.*

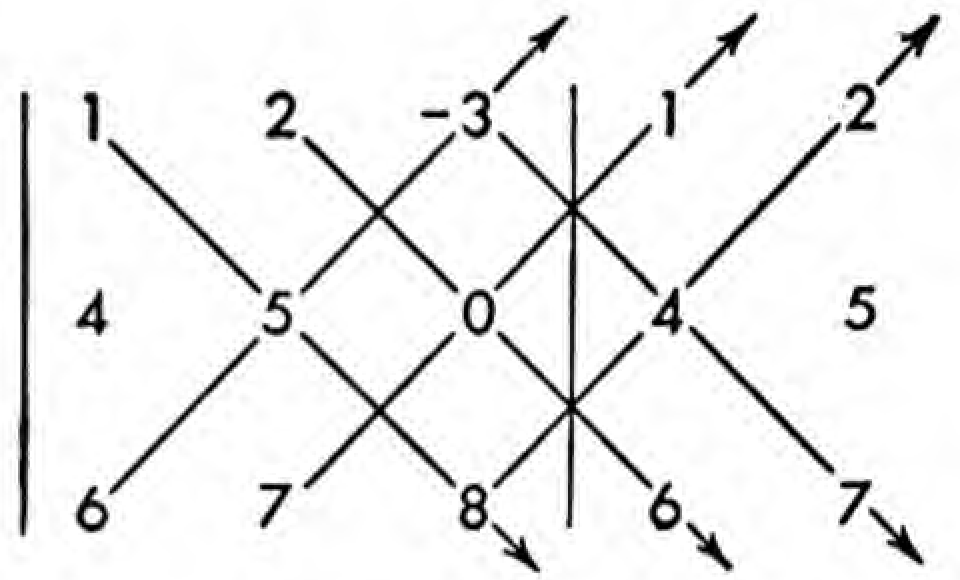


Illustration 1.

$$\begin{vmatrix} 1 & 2 & -3 \\ 4 & 5 & 0 \\ 6 & 7 & 8 \end{vmatrix}$$

$$= 1\cdot5\cdot8 + 2\cdot0\cdot6 + (-3)\cdot4\cdot7 - 6\cdot5\cdot(-3) - 7\cdot0\cdot1 - 8\cdot4\cdot2$$
$$= 40 + 0 - 84 + 90 - 0 - 64 = -18.$$

45. Solution of three linear equations by determinants. Every system of three linear equations in x, y, and z can be written in the form

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3, \end{cases}$$

where the a's, b's, c's, and d's are any constants. If the method of Art. 41 is applied, we get

$$x = \frac{d_1b_2c_3 + d_3b_1c_2 + d_2b_3c_1 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3},$$

provided the denominator is not zero.

We notice that the denominator is the expansion of the determinant of the coefficients. The numerator can be obtained from

the denominator if the a's are replaced by the corresponding d's. Hence we can write

$$x = \frac{\begin{vmatrix} \mathbf{d}_1 & b_1 & c_1 \\ \mathbf{d}_2 & b_2 & c_2 \\ \mathbf{d}_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Using the same method, we find that

$$y = \frac{\begin{vmatrix} a_1 & \mathbf{d}_1 & c_1 \\ a_2 & \mathbf{d}_2 & c_2 \\ a_3 & \mathbf{d}_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & \mathbf{d}_1 \\ a_2 & b_2 & \mathbf{d}_2 \\ a_3 & b_3 & \mathbf{d}_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

We conclude that Cramer's rule can be used in solving a system of three linear equations in three unknowns.

Example 1. Solve by determinants: $\begin{cases} 2x + 3y + 6z = 1, \\ x + y - z = 0, \\ 5x + 2y + z = 2. \end{cases}$

Solution. Apply Cramer's rule.

$$x = \frac{\begin{vmatrix} 1 & 3 & 6 \\ 0 & 1 & -1 \\ 2 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 6 \\ 1 & 1 & -1 \\ 5 & 2 & 1 \end{vmatrix}} = \frac{1 - 6 + 0 - 12 + 2 - 0}{2 - 15 + 12 - 30 + 4 - 3} = \frac{-15}{-30} = \frac{1}{2}.$$

$$y = \frac{\begin{vmatrix} 2 & 1 & 6 \\ 1 & 0 & -1 \\ 5 & 2 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 6 \\ 1 & 1 & -1 \\ 5 & 2 & 1 \end{vmatrix}} = \frac{10}{-30} = -\frac{1}{3}, \qquad z = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 1 & 1 & 0 \\ 5 & 2 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 6 \\ 1 & 1 & -1 \\ 5 & 2 & 1 \end{vmatrix}} = \frac{-5}{-30} = \frac{1}{6}.$$

Exercise 25

Solve by use of determinants.

1. $\begin{cases} x + y + z = 5, \\ 2x + 3y - 4z = 0, \\ 5x + 6y - 7z = 3. \end{cases}$

2. $\begin{cases} 2x - y - z = 1, \\ 3x - 4y - 2z = 5, \\ x - 2y - 3z = -4. \end{cases}$

3. $\begin{cases} x + y + 2z = 1, \\ 4x - y + 2z = 2, \\ x - 3y - 6z = -1. \end{cases}$

4. $\begin{cases} 3x + 4y + 5z = -1, \\ 2x - 3y = 2, \\ 3x + 2y + 4z = 0. \end{cases}$

5. $\begin{cases} 2x + 3y + z = 1, \\ 3x - 4y - 5z = -4, \\ 4x + 5y - 3z = 1. \end{cases}$

6. $\begin{cases} 5x - y + 2z = 2, \\ 4x - 2y + 3z = 2, \\ 3x - 3y + 5z = 3. \end{cases}$

Problems 7 to 14. Solve problems 7 to 14 in Exercise 22 by using determinants.

chapter 7 *Exponents and radicals*

46. Positive integral exponents. In Art. 9 we formulated the following definition for the case in which m is a positive integer:

$$a^m = a \cdot a \cdot a \cdots \text{to } m \text{ factors.}$$

The symbol a^m is read "the mth power of a" or "a to the mth." We call a the **base** and m the **exponent.**

As a consequence of this definition, we can prove the following *Laws of Exponents.* It is understood that all exponents are positive integers.

$$[1] \qquad a^m \cdot a^n = a^{m+n}.$$

To multiply two powers of the same base, hold the base and add the exponents.

$$[2] \qquad (a^m)^n = a^{mn}.$$

To find a power of a power of a certain base, hold the base and multiply the exponents.

$$[3a] \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (m > n).$$

$$[3b] \qquad \frac{a^m}{a^n} = \frac{1}{a^{n-m}} \qquad (m < n).$$

$$[4] \qquad (ab)^n = a^n b^n.$$

$$[5] \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Proofs. For [1]

$$\begin{aligned} a^m \cdot a^n &= (a \cdot a \cdot a \cdots \text{to } m \text{ factors})(a \cdot a \cdot a \cdots \text{to } n \text{ factors}) \\ &= a \cdot a \cdot a \cdots \text{to } (m+n) \text{ factors} = a^{m+n}. \end{aligned}$$

For [2]

$$\begin{aligned} (a^m)^n &= a^m \cdot a^m \cdot a^m \cdots \text{to } n \text{ factors} \\ &= a^{m+m+m+ \cdots \text{ to } n \text{ terms}} = a^{mn}. \end{aligned}$$

For [3*a*]

$$\begin{aligned} \frac{a^m}{a^n} &= \frac{\cancel{a} \cdot \cancel{a} \cdots \cancel{a} \cdot a \cdots a}{\cancel{a} \cdot \cancel{a} \cdots \cancel{a}} \qquad \begin{matrix} (m \text{ factors}) \\ (n \text{ factors}) \end{matrix} \\ &= a \cdot a \cdot a \cdots \text{to } (m-n) \text{ factors} = a^{m-n}. \end{aligned}$$

For [4]

$$\begin{aligned} (ab)^n &= (ab)(ab)(ab) \cdots \text{to } n \text{ factors} \\ &= (a \cdot a \cdot a \cdots \text{to } n \text{ factors})(b \cdot b \cdot b \cdots \text{to } n \text{ factors}) \\ &= a^n b^n. \end{aligned}$$

For [5]

$$\begin{aligned} \left(\frac{a}{b}\right)^n &= \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \text{to } n \text{ factors} \\ &= \frac{a \cdot a \cdot a \cdots \text{to } n \text{ factors}}{b \cdot b \cdot b \cdots \text{to } n \text{ factors}} = \frac{a^n}{b^n}. \end{aligned}$$

It is important to guard against the common habit of inserting parentheses mentally when there is no justification for this. For example, -7^2 means "the negative of the square of 7," which is -49. But $(-7)^2$ means "the square of -7," which is 49.

Exercise 26

Evaluate.

1. -3^2.	**2.** $(-3)^2$.	**3.** $(-2)^3$.	**4.** -2^3.
5. $\frac{1}{2} \cdot 4^3$.	**6.** $(\frac{1}{2} \cdot 4)^3$.	**7.** $(-\frac{3}{5})^2$.	**8.** $(-\frac{2}{3})^4$.

Perform the indicated operations by using the laws of exponents.

9. a^2a^3.	**10.** a^4a^k.	**11.** t^2tt^5.	**12.** $2^{m+1}2^{3-m}$.
13. $(a^2)^3$.	**14.** $(x^7)^4$.	**15.** $(rs^5)^2$.	**16.** $(2b)^5$.
17. $(3xy^3)^4$.	**18.** $(.3w^2z^3)^2$.	**19.** $(-5c^3d^6)^3$.	**20.** $(-.1x^5y^4)^4$.

21. $\frac{a^6}{a^2}$. 22. $\frac{x^{12}}{x^4}$. 23. $\frac{t^4}{t^8}$. 24. $\frac{a^3}{a^9}$.

25. $\frac{x^3y^{10}}{x^4y^8}$. 26. $\frac{r^6s^5}{r^4s^9}$. 27. $\frac{10^{m+6}}{10^{m+3}}$. 28. $\frac{7^{t+3}}{7^{t+1}}$.

29. $\left(\frac{x^3}{y^5}\right)^2$. 30. $\left(\frac{4}{c^4}\right)^3$. 31. $\left(\frac{5rs^2}{t^4}\right)^3$. 32. $\left(-\frac{x^5y}{10z^4}\right)^5$.

33. $\left(-\frac{x^3y^4}{2z}\right)^4$. 34. $\left(\frac{a^pb^2}{c^n}\right)^n$.

35. $\left(\frac{x^2}{5y^4}\right)^3\left(\frac{10y}{x^3}\right)^2$. 36. $(x+y)^{100}(x+y)^3$.

37. Compute the value of $2x^3$ when $x = 5$.

38. Compute the value of $-3x^2$ when $x = 2$.

47. Roots. We say that A is a square root of B if $A^2 = B$. Every positive number has two real square roots, one positive and the other negative. For example, the square roots of 49 are 7 and -7 because $7^2 = 49$ and $(-7)^2 = 49$.

If a certain number is negative, it has no real square root because the square of a real number (positive, negative, or zero) cannot be negative. In Art. 59, we shall show that every negative number has two square roots which are called imaginary numbers — to distinguish them from real numbers. For the present, when we encounter a square root of a negative number we shall merely call it an imaginary number, i.e., it is not positive, negative, or zero.

We call A a cube root of B if $A^3 = B$. If n is any positive integer, we say that

A is an nth root of B if $A^n = B$.

Thus -2 is a cube root of -8 because $(-2)^3 = -8$. And 5 is a fourth root of 625 because $5^4 = 625$.

In Chapter 15 we shall prove the following statements.

Every number (except zero) has n distinct nth roots, some or all of which may be imaginary.

If n is *odd*, every real number B has only one real nth root, which is positive when B is positive, and negative when B is negative. Thus the real cube root of 125 is 5; the real fifth root of -32 is -2.

If n is *even*, every *positive* number B has exactly two real nth roots,

one positive and one negative, with equal absolute values. Thus the two real sixth roots of 64 are 2 and -2.

If n is *even* and B is *negative*, all the nth roots of B are imaginary numbers. Thus the two square roots of -81 are imaginary numbers.

The **principal nth root** of B is defined to be

(1) the positive nth root of B if B is positive,

(2) the negative nth root of B if B is negative and n is odd.

We do not define the principal nth root of B if B is negative and n is even.

Illustration 1. The principal square root of 49 is 7. The principal cube root of -125 is -5. All sixth roots of -1 are imaginary numbers.

48. Radicals. We shall use the symbol $\sqrt[n]{a}$ to represent *the principal* nth root of a.* The integer n is called the **order** (or **index**) of the **radical** $\sqrt[n]{a}$, while a is called the **radicand.** If the index is 2, we write $\sqrt{a}$ rather than $\sqrt[2]{a}$ to indicate the square root of a.

Illustration 1. $\sqrt[4]{10{,}000} = 10$. $\sqrt[3]{8} = 2$. $\sqrt[3]{-8} = -2$. $\sqrt{-100}$ is imaginary.

Illustration 2. Since the symbol $\sqrt{25}$ means *the principal* square root of 25, it can represent only one number. Thus $\sqrt{25} = 5$, and not -5. But $-\sqrt{25} = -5$, and $\pm\sqrt{25} = \pm 5$. Hence the two square roots of 25 are $\sqrt{25}$ and $-\sqrt{25}$, or 5 and -5.

By the definition of an nth root, we have

$$(\sqrt[n]{a})^n = a \qquad \text{and} \qquad \sqrt[n]{a^n} = a.$$

Exercise 27

State the two square roots of each number.

1. 121. **2.** 1. **3.** $\frac{4}{25}$. **4.** .09.

State the principal square root of each number.

5. .36. **6.** $\frac{1}{64}$. **7.** 81. **8.** 16.

* To avoid ambiguity and confusion, we shall restrict ourselves to positive values of a when n is even.

State the principal cube root of each number.

9. -125. 10. $-\frac{1}{8}$. 11. $\frac{8}{27}$. 12. 216.

Find, by inspection, the value of each radical.

13. $\sqrt{144}$. 14. $\sqrt[3]{1000}$. 15. $\sqrt[3]{-27}$. 16. $\sqrt[4]{81}$.
17. $\sqrt[5]{\frac{1}{32}}$. 18. $\sqrt[5]{-1}$. 19. $\sqrt[4]{.0016}$. 20. $\sqrt{\frac{25}{49}}$.
21. $-\sqrt[3]{x^3}$. 22. $-\sqrt{t^2}$. 23. $\sqrt{16y^{16}}$. 24. $\sqrt[3]{8y^{12}}$.

Find the value of each power.

25. $(\sqrt{1865})^2$. 26. $(\sqrt[3]{-1492})^3$.
27. $(-\sqrt[3]{1776})^3$. 28. $(-\sqrt{1914})^2$.
29. $(-\sqrt[4]{1066})^4$. 30. $(-\sqrt[5]{1620})^5$.
31. $(\sqrt[3]{-7ab^2})^3$. 32. $(-\sqrt[9]{-5x^6y^7})^9$.

49. Fractional exponents. In Art. 46 we defined the symbol a^m for the case in which m is a *positive integer*. As a consequence of this definition, we were able to prove the five laws of exponents.

At this stage of our discussion, a fractional exponent has no meaning. For the sake of uniformity we shall define it in such a way that all fractional exponents shall behave exactly as do the positive integral exponents, i.e., in accordance with the five laws.

If $a^{\frac{1}{2}}$ is to obey law [1], we have

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

Hence $a^{\frac{1}{2}}$ is a quantity which when multiplied by itself is equal to a. Therefore $a^{\frac{1}{2}}$ must be the * square root of a:

$$a^{\frac{1}{2}} = \sqrt{a}.$$

In general, $a^{\frac{1}{n}}$ must be the * nth root of a:

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

If $a^{\frac{2}{3}}$ is to obey law [1],

$$a^{\frac{2}{3}}a^{\frac{2}{3}}a^{\frac{2}{3}} = a^{\frac{2}{3}+\frac{2}{3}+\frac{2}{3}} = a^2.$$

* We define it to be the principal root.

Since $a^{\frac{2}{3}}$ is one of the three equal factors of a^2, it must represent the cube root of a^2:

$$a^{\frac{2}{3}} = \sqrt[3]{a^2}.$$

All fractional exponents will obey law [1] if we define *

$$[6a] \qquad a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

$$[6b] \qquad a^{\frac{m}{n}} = \sqrt[n]{a^m},$$

where m and n are positive integers. Either of these forms may be used but in some cases one of them is more convenient than the other. The symbol $(\sqrt[n]{a})^m$ is read "the mth power of the nth root of a." The symbol $\sqrt[n]{a^m}$ is read "the nth root of a to the mth."

Illustration 1. $a^{\frac{1}{7}} = \sqrt[7]{a}$. $32^{\frac{1}{5}} = \sqrt[5]{32} = 2$. $a^{\frac{5}{8}} = \sqrt[8]{a^5}$, using [6b].

Illustration 2. $25^{\frac{3}{2}} = (\sqrt{25})^3 = 5^3 = 125$, using [6a], which is preferable in some numerical evaluations. Notice the involved computation if [6b] is used: $25^{\frac{3}{2}} = \sqrt{25^3} = \sqrt{15625} = 125$.

It is possible to show that definition [6a] is consistent with all five laws of exponents. We shall assume without proof that these laws hold for all positive rational exponents.

Illustration 3. $\left(\dfrac{16x^{16}}{y^{40}}\right)^{\frac{3}{4}} = \dfrac{16^{\frac{3}{4}}(x^{16})^{\frac{3}{4}}}{(y^{40})^{\frac{3}{4}}} = \dfrac{(\sqrt[4]{16})^3 x^{16(\frac{3}{4})}}{y^{40(\frac{3}{4})}} = \dfrac{8x^{12}}{y^{30}}.$

50. Zero as an exponent. If a^0 is to obey law [1],

$$a^0a^n = a^{0+n} = a^n, \qquad \text{or} \qquad a^0a^n = a^n.$$

Divide through by a^n: $\qquad a^0 = \dfrac{a^n}{a^n} = 1.$

Another explanation: since a^n is unchanged when multiplied by a^0, we must conclude that

$$[7] \qquad a^0 = 1 \qquad (a \neq 0).$$

This means that the zero*th* power of any number a is equal to 1, provided a is not zero. We shall attach no meaning to the symbol 0^0.

Illustration 1. $5^0 = 1$. $(-\frac{2}{3})^0 = 1$. $6(\frac{3}{4})^0 = 6 \cdot 1 = 6$. $7x^0 = 7$.

* The two definitions are equivalent because $(\sqrt[n]{a})^m = (a^{\frac{1}{n}})^m = a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$. We rule out the case in which a is negative and n is even.

51. Negative exponents. If a^{-n} is to obey law [1],

$$a^{-n}a^n = a^{-n+n} = a^0 = 1.$$

$$a^{-n}a^n = 1.$$

Divide through by a^n:

[8] $$a^{-n} = \frac{1}{a^n}.$$

This means that a^{-n} is the reciprocal of a^n. Moreover, a negative power of a fraction is equal to the corresponding positive power of the reciprocal of the fraction:

$$\left(\frac{c}{d}\right)^{-n} = \left(\frac{d}{c}\right)^n.$$

Illustration 1.

$$2^{-6} = \frac{1}{2^6} = \frac{1}{64}. \quad 3a^{-2} = 3\left(\frac{1}{a^2}\right) = \frac{3}{a^2}. \quad 4^{-\frac{3}{2}} = \frac{1}{4^{\frac{3}{2}}} = \frac{1}{(\sqrt{4})^3} = \frac{1}{2^3} = \frac{1}{8}.$$

Illustration 2. $(\frac{5}{2})^{-3} = (\frac{2}{5})^3 = \frac{8}{125}.$

It follows from [8] that *in a fraction, any* **factor** *of the numerator may be transferred to the denominator* (*or vice versa*) *if the sign of the exponent of the factor is changed.*

Illustration 3. $\dfrac{7x^{-1}y^5}{s^3t^{-4}} = \dfrac{7t^4y^5}{s^3x}.$

Example 1. Simplify (rewrite without negative exponents and reduce to a simple fraction): $\dfrac{5a^{-1}}{a^{-2} + b^{-3}}.$

Solution 1.

$$\frac{5a^{-1}}{a^{-2}+b^{-3}} = \frac{5\left(\frac{1}{a}\right)}{\frac{1}{a^2}+\frac{1}{b^3}} = \frac{\frac{5}{a}}{\frac{b^3+a^2}{a^2b^3}} = \frac{5}{a}\cdot\frac{a^2b^3}{b^3+a^2} = \frac{5ab^3}{b^3+a^2}.$$

Solution 2. We see by inspection that negative exponents can be eliminated by multiplying top and bottom by a^2b^3. This gives

$$\frac{5a^{-1}a^2b^3}{(a^{-2}+b^{-3})a^2b^3} = \frac{5ab^3}{b^3+a^2}.$$

Incorrect solution. $\dfrac{5a^{-1}}{a^{-2}+b^{-3}} = \dfrac{5(a^2+b^3)}{a}.$ Since a^{-2} is not a *factor* of the denominator, the foregoing principle does not apply to the given fraction.

Example 2. Simplify $\left(\dfrac{3a^{-1}b^{\frac{5}{4}}}{2b^{-\frac{3}{4}}c^7}\right)^{-3}$.

Solution. $\left(\dfrac{3a^{-1}b^{\frac{5}{4}}}{2b^{-\frac{3}{4}}c^7}\right)^{-3} = \left(\dfrac{3b^{\frac{3}{4}}b^{\frac{5}{4}}}{2ac^7}\right)^{-3} = \left(\dfrac{2ac^7}{3b^2}\right)^3 = \dfrac{8a^3c^{21}}{27b^6}$.

Exercise 28

Evaluate.

1. $49^{\frac{1}{2}}$. **2.** $8^{\frac{1}{3}}$. **3.** $(\frac{1}{32})^{\frac{1}{5}}$. **4.** $(\frac{1}{81})^{\frac{1}{4}}$.

5. $27^{\frac{2}{3}}$. **6.** $32^{\frac{3}{5}}$. **7.** $100^{\frac{3}{2}}$. **8.** $(1{,}000{,}000)^{\frac{5}{6}}$.

9. $(\frac{1}{4})^{\frac{5}{2}}$. **10.** $(.125)^{\frac{4}{3}}$. **11.** $(\frac{8}{27})^{\frac{1}{3}}$. **12.** $(\frac{4}{9})^{\frac{1}{2}}$.

13. $8(-\frac{3}{4})^0$. **14.** $(-7)^0$. **15.** $4b^0$. **16.** $-6(\frac{2}{3})^0$.

17. $(9^{\frac{1}{2m}})^m$. **18.** $(8^{\frac{r}{3}})^{\frac{2}{r}}$. **19.** 7^{-1}. **20.** 5^{-2}.

21. $(-3)^{-4}$. **22.** $(-4)^{-3}$. **23.** $(\frac{1}{2})^{-3}$. **24.** $(\frac{3}{7})^{-1}$.

25. $16^{-\frac{3}{4}}$. **26.** $(-1000)^{-\frac{2}{3}}$. **27.** $81^{-\frac{1}{2}}$. **28.** $4^{-\frac{5}{2}}$.

29. $(\frac{1}{4})^{-\frac{3}{2}}$. **30.** $(\frac{1}{10{,}000})^{-\frac{3}{4}}$. **31.** $(\frac{27}{8})^{-\frac{1}{3}}$. **32.** $(\frac{25}{49})^{-\frac{1}{2}}$.

33. $27 - 18 \cdot 3^{-2}$. **34.** $\dfrac{5}{10^{-4}}$.

Express without negative exponents.

35. x^{-3}. **36.** b^{-5}. **37.** $4a^{-1}$. **38.** $6y^{-2}$.

39. $\dfrac{a^2}{b^{-6}}$. **40.** $\dfrac{x^{-5}}{y^6}$. **41.** $\dfrac{s^{-2}}{rt^{-1}}$. **42.** $\dfrac{b^{-7}}{ac^{-8}}$.

43. $\dfrac{10}{(2x)^{-3}}$. **44.** $\dfrac{2a^{-1}b^{-2}}{3a^{-2}c}$.

Write without a denominator by using negative exponents.

45. $\dfrac{5}{x^3}$. **46.** $\dfrac{W^2}{(1.02)^3}$. **47.** $\dfrac{7x}{ab^3}$. **48.** $\dfrac{a}{b+c}$.

Write without fractional exponents by using radicals.

49. $a^{\frac{2}{7}}$. **50.** $x^{\frac{3}{8}}$. **51.** $(10x)^{\frac{2}{3}}$.

52. $7(cy)^{\frac{5}{6}}$. **53.** $5a^{\frac{1}{2}}$. **54.** $bx^{\frac{4}{5}}$.

Write without radicals by using fractional exponents.

55. $\sqrt[7]{a^5}$. **56.** $\sqrt[10]{x^6}$. **57.** $\sqrt[5]{(x+y)^4}$. **58.** $3\sqrt[4]{a-c}$.

Perform the indicated operations and simplify. (Write without negative or zero exponents. Reduce fractions to lowest terms.)

59. $x^{\frac{1}{3}}x$. 60. y^0y^6. 61. 2^t2^{1-t}. 62. $10^{4s+2}10^{1-4s}$.

63. $\dfrac{x^2}{x^{\frac{6}{5}}}$. 64. $\dfrac{a}{a^{\frac{7}{4}}}$. 65. $\dfrac{ab^{\frac{2}{3}}}{a^{\frac{1}{2}}b}$. 66. $\dfrac{x^{\frac{7}{5}}y^3}{x^2y^{\frac{9}{4}}}$.

67. $\dfrac{20x^{\frac{1}{2}}}{5x^{\frac{1}{3}}}$. 68. $\dfrac{14y^{\frac{3}{4}}}{21y^{\frac{2}{3}}}$. 69. $(x^{\frac{2}{3}})^{27}$. 70. $(y^{\frac{3}{4}})^{20}$.

71. $(3x^{-5})^2$. 72. $(10x^{-1}y^2)^3$. 73. $(16z^4b^8)^{\frac{1}{4}}$. 74. $(64x^{12}y^{30})^{\frac{2}{3}}$.

75. $(a^3b^4)^{-5}$. 76. $(r^{-1}s^{-3})^{-2}$. 77. $(2a^{-4}b)^{-3}$. 78. $(5xy^{-8})^{-4}$.

79. $(4a^{-4})^{-\frac{5}{2}}$. 80. $(49x^{-6}y^0)^{-\frac{1}{2}}$. 81. $(125x^{-6})^{-\frac{2}{3}}$. 82. $(32a^{-10})^{-\frac{4}{5}}$.

83. $\dfrac{12x^{-1}y^2}{16x^3y^{-4}}$. 84. $\dfrac{33ab^{-7}}{55a^3b^{-4}}$. 85. $\dfrac{24a^2b^{-3}}{42a^6b^{-6}}$. 86. $\dfrac{2a^{-1}b^{-2}}{8a^{-5}b^3}$.

87. $(3^{-1} + 6^{-1})^{-2}$. 88. $\dfrac{6^{-1} + 3^{-2}}{2^{-3}}$. 89. $\dfrac{7a^{-2}}{(3a)^{-2} + 5b^{-1}}$.

90. $(2a^{-4} + 6^{-1})^{-1}$. 91. $\dfrac{7a^{-1}b^{-2}}{a^{-3} - 5^0}$. 92. $\dfrac{x^{-1} + y^{-1}}{(x + y)^{-1}}$.

93. $\dfrac{(\frac{1}{2})^{-3} - x^{-2}}{4x^{-1}}$. 94. $x^{-\frac{1}{2}} + x^{-\frac{3}{2}}$. 95. $\left(\dfrac{7a^{\frac{1}{2}}b^{-1}}{9a^{\frac{1}{4}}b}\right)^2$.

96. $\left(\dfrac{2x^2y^{\frac{1}{3}}}{5x^{-2}y}\right)^3$. 97. $\left(\dfrac{x^{-\frac{4}{5}}y^2}{2x^{\frac{1}{5}}y^{-5}}\right)^{-3}$. 98. $\left(\dfrac{8a^{-\frac{1}{3}}b}{3a^{\frac{5}{3}}b^{-4}}\right)^{-2}$.

99. $\left(\dfrac{8x^{-4}y^8}{27x^5y^2}\right)^{\frac{2}{3}}$. 100. $\left(\dfrac{9a^3b^{-2}}{25a^{-1}b^4}\right)^{\frac{3}{2}}$. 101. $\left(\dfrac{25a^{\frac{1}{4}}b^{-7}}{36a^{\frac{9}{4}}b^9}\right)^{-\frac{1}{2}}$.

102. $\left(\dfrac{81x^{-3}y^{10}}{16x^5y^{-2}}\right)^{-\frac{3}{4}}$. 103. $\left(\dfrac{7a^2}{b^n}\right)^n$. 104. $(2x^{-r}y^{2t})^t$.

Expand and then simplify without eliminating negative exponents.

105. $(e^x + e^{-x})^2$. 106. $(5x + y^{-3})(5x - y^{-3})$.

107. $(a^{\frac{1}{2}} - 3b^{\frac{1}{2}})(a^{\frac{1}{2}} - 2b^{\frac{1}{2}})$. 108. $(a^{\frac{1}{2}} - 7b^{\frac{1}{2}})^2$.

109. $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$. 110. $(a^{\frac{1}{3}} + 2b^{\frac{1}{3}})(a^{\frac{2}{3}} - 2a^{\frac{1}{3}}b^{\frac{1}{3}} + 4b^{\frac{2}{3}})$.

Perform the following long divisions without eliminating negative exponents.

111. $(6x^{-3} + 14x^{-2} + 19x^{-1} + 5) \div (3x^{-1} + 1)$.

112. $(2x^{-3} + x^{-2} + 3x^{-1} - 1 + 4x) \div (x^{-2} - x^{-1} + 1)$.

113. $(x^{\frac{5}{4}} - 3x^{\frac{1}{4}} + x^{-\frac{3}{4}}) \div (x^{\frac{1}{2}} + 1 - x^{-\frac{1}{2}})$.

114. $(40x^2 + 21x^{\frac{3}{2}} - 27x + 5x^{\frac{1}{2}} - 3) \div (5x^{\frac{1}{2}} - 3)$.

115. Write as a power of 2:

(a) $2^n + 2^n$. (b) $8 \cdot 4^n$. (c) $2 \cdot 16^{\frac{n}{2}}$.

116. Identify as true or false and give reasons:

(a) $3^n 5^{2n} = 75^n$. (b) $(a^3 + a^4)^2 = a^6 + a^8$. (c) $a^4 - a^2 = a^2$.

Express each of the following numbers as an integer times a power of 10.

117. 0.000 000 17.

Solution. $0.000\,000\,17 = \dfrac{17}{100{,}000{,}000} = \dfrac{17}{10^8} = 17(10^{-8})$.

118. 0.000 000 009.

119. 0.000 000 000 04.

120. 93,000,000.

121. 25,000,000,000,000.

Write in ordinary notation.

122. $2.6(10^{11})$. **123.** $67(10^{12})$. **124.** $48(10^{-10})$. **125.** $1.6(10^{-14})$.

52. Properties of radicals. The form of a radical can be changed (without altering its value) by use of the following properties.*

Property 1. $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$,

i.e. *the* n*th root of a product is equal to the product of the* n*th roots of its factors.*

Illustration 1. $\sqrt[3]{-8a^{15}} = \sqrt[3]{-8}\,\sqrt[3]{a^{15}} = -2a^5$.

A most common mistake is to assume that property 1 applies to the sum of two numbers. It is not generally true that $\sqrt[n]{a+b}$ equals $\sqrt[n]{a} + \sqrt[n]{b}$. To show this, let $n = 2$, $a = 9$, $b = 16$. Then $\sqrt{9+16} = \sqrt{25} = 5$, whereas $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$.

Property 2. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Illustration 2. $\sqrt[4]{\dfrac{x^4y^{16}}{81z^{40}}} = \dfrac{\sqrt[4]{x^4y^{16}}}{\sqrt[4]{81z^{40}}} = \dfrac{xy^4}{3z^{10}}$.

Property 3. $\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}}$.

This is read "the mnth root of a is equal to the mth root of the nth root of a," etc.

* Again we restrict ourselves to a positive radicand when the order is even.

Illustration 3. $\sqrt[6]{125} = \sqrt{\sqrt[3]{125}} = \sqrt{5}.$

Illustration 4. $\sqrt[3]{\sqrt[4]{a^3}} = \sqrt[4]{\sqrt[3]{a^3}} = \sqrt[4]{a}.$

These three properties can be proved by converting the radicals to expressions involving fractional exponents and then applying the laws of exponents. For property 1,

$$\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}} = \sqrt[n]{a}\,\sqrt[n]{b}.$$

For property 2,

$$\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

For property 3,

$$\sqrt[mn]{a} = a^{\frac{1}{mn}} = a^{\frac{1}{n}\cdot\frac{1}{m}} = (a^{\frac{1}{n}})^{\frac{1}{m}} = \sqrt[m]{\sqrt[n]{a}}, \text{ etc.}$$

53. Simplifying a radical. If a radical of order n contains a radicand that is a perfect nth power, we can simplify the radical by using the relation $\sqrt[n]{a^n} = a$. In this case the expression is said to be **rational.** If the radical sign cannot be removed, we simplify the radical by performing the following operations whenever possible.

(1) **Reducing the order.** This is accomplished by using property 3:

$$\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}.$$

Illustration 1. $\sqrt[6]{49x^2y^4} = \sqrt[3]{\sqrt{49x^2y^4}} = \sqrt[3]{7xy^2}.$

(2) **Removing factors from the radicand.** In the case of a radical of order n, express the radicand as the product of two factors, one of which is a perfect nth power. Then remove this factor from the radicand by using property 1.

Illustration 2.

$$\sqrt{63a^3y^4} = \sqrt{9a^2y^4 \cdot 7a} = \sqrt{9a^2y^4}\sqrt{7a} = 3ay^2\sqrt{7a}.$$
$$\sqrt[3]{-40a^8} = \sqrt[3]{-8a^6 \cdot 5a^2} = -2a^2\sqrt[3]{5a^2}.$$

(3) **Rationalizing the denominator.** It is usually desirable to convert a fractional radicand to one that is integral. In a radical of order n, multiply the numerator and denominator of the radicand by some quantity that will make the denominator a perfect nth power. Then apply property 2.

Illustration 3.

$$\sqrt{\frac{3x}{7y}} = \sqrt{\frac{3x}{7y}\cdot\frac{7y}{7y}} = \sqrt{\frac{21xy}{49y^2}} = \frac{\sqrt{21xy}}{\sqrt{49y^2}} = \frac{\sqrt{21xy}}{7y}\left(\text{or } \frac{1}{7y}\sqrt{21xy}\right).$$

$$\sqrt[4]{\frac{a}{8xy^2}} = \sqrt[4]{\frac{a}{8xy^2}\cdot\frac{2x^3y^2}{2x^3y^2}} = \sqrt[4]{\frac{2ax^3y^2}{16x^4y^4}} = \frac{1}{2xy}\sqrt[4]{2ax^3y^2}.$$

If large numbers are involved, it is wise to find the prime factors of these numbers before determining the quantity by which we multiply numerator and denominator.*

Illustration 4.

$$\sqrt[3]{\frac{135a^{14}}{784x^8}} = \sqrt[3]{\frac{3^3\cdot 5a^{14}}{2^4\cdot 7^2\cdot x^8}\cdot\frac{2^2\cdot 7\cdot x}{2^2\cdot 7\cdot x}} = \sqrt[3]{\frac{3^3a^{12}}{2^6 7^3 x^9}\cdot 5\cdot 2^2\cdot 7\cdot a^2x}$$
$$= \frac{3a^4}{28x^3}\sqrt[3]{140a^2x}.$$

Exercise 29

Reduce the order of each radical.

1. $\sqrt[4]{9}$.
2. $\sqrt[6]{25}$.
3. $\sqrt[10]{x^6}$.
4. $\sqrt[15]{32}$.
5. $\sqrt[6]{1000x^3}$.
6. $\sqrt[8]{81y^6}$.
7. $\sqrt[8]{x^2+2x+1}$.
8. $\sqrt[4]{\frac{100}{t^2}}$.

Remove factors from radicand.

9. $\sqrt{20}$.
10. $\sqrt{18}$.
11. $\sqrt{16y^{16}}$.
12. $\sqrt{72x^2}$.
13. $\sqrt[3]{56}$.
14. $\sqrt[3]{27y^{27}}$.
15. $\sqrt[3]{-54x^7}$.
16. $\sqrt[3]{-48x^4}$.
17. $\sqrt[4]{48x^5y^7}$.
18. $\sqrt[4]{80x^6y^9}$.
19. $\sqrt{(x^2-y^2)(x+y)}$.
20. $\sqrt{(x+y)(x^3+y^3)}$.
21. $\sqrt{4x^2+16}$.
22. $\sqrt[3]{8x^3+64}$.
23. $\sqrt[3]{\frac{r^4s^5}{8t^6}}$.
24. $\sqrt{\frac{8a^3b^7}{81x^6}}$.

* See Ex. 2, Art. 18.

Rationalize denominators.

25. $\sqrt{\frac{2}{3}}$. 26. $\sqrt{\frac{3}{10}}$. 27. $\sqrt{\frac{7}{5x}}$. 28. $\sqrt{\frac{5}{2a}}$.

29. $\sqrt[3]{\frac{a}{9x^2y}}$. 30. $\sqrt[3]{\frac{2a}{7bc^2}}$. 31. $\sqrt[4]{\frac{4}{xy^2z^3}}$. 32. $\sqrt{\frac{3}{x+y}}$.

Simplify.

33. $\sqrt{\frac{4a}{9b^2}}$. 34. $\sqrt{\frac{40x^6}{y^8}}$. 35. $\sqrt{\frac{49x^4}{2}}$. 36. $\sqrt{\frac{700a^2}{b}}$.

37. $\sqrt{\frac{44x^{64}}{y^7}}$. 38. $\sqrt{\frac{54s^8}{t^3}}$. 39. $\sqrt[3]{\frac{-125}{ab^5}}$. 40. $\sqrt[3]{\frac{-64x}{y^2z^8}}$.

41. $\sqrt[3]{\frac{a^8b}{9b^7}}$. 42. $\sqrt[3]{\frac{a^4b^5}{5b^2}}$. 43. $\sqrt[4]{\frac{x^5y^7}{8z}}$ 44. $\sqrt[5]{\frac{a^8b^{10}}{4}}$.

45. $\sqrt{\frac{28x^3}{45y^4z^5}}$. 46. $\sqrt{\frac{27x^7}{32y^3z^6}}$. 47. $\sqrt[3]{8a^3+8b^{-3}}$. 48. $\sqrt{9x^{-2}+9y^4}$.

49. $\sqrt[6]{100x^{10}}$. 50. $\sqrt[4]{121x^6}$. 51. $\sqrt{\frac{1}{4}+\frac{1}{16}}$. 52. $\sqrt{\frac{a}{b}+\frac{c}{d}}$.

53. $\sqrt{\frac{9x}{x+y^2}}$. 54. $\sqrt{\frac{a}{y^2}+\frac{2a}{xy}+\frac{a}{x^2}}$. 55. $\sqrt[3]{(a+b)^2(a^2-b^2)}$.

56. $\sqrt[t]{\frac{x^t}{y^{t-1}}}$. 57. $\sqrt[n]{a^{nt}b^{2n+1}}$. 58. $\sqrt[n]{x^{3n}y^{n+2}}$.

59. $\sqrt{\frac{16}{25a}-1}$. 60. $\sqrt[3]{\frac{1}{4}+a^{-3}}$. 61. $\sqrt[3]{\frac{88a^{-4}b^{13}}{a^3b^{-14}}}$.

62. $\sqrt{\frac{16x^{-11}y^{-1}}{25x^5z^{-7}}}$. 63. $\sqrt{\frac{176a^{49}}{525b^{20}}}$. 64. $\sqrt[3]{\frac{686x^{64}}{2025y^9}}$.

Simplify and then compute to three-decimal accuracy by use of Table I.

65. $\sqrt{630}$. 66. $\sqrt{175}$. 67. $\sqrt{\frac{64}{7}}$. 68. $\sqrt{\frac{7}{12}}$.

69. $\sqrt[3]{.07}$. 70. $\sqrt[3]{\frac{11}{24}}$. 71. $\sqrt[6]{25}$. 72. $\sqrt[8]{81}$.

73. The area of a triangle with sides a, b, c, is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. Use this formula to find the area of a triangle with sides 5, 10, 11.

54. Addition and subtraction of radicals. Two radicals can be added only if they are of the *same order* and have the *same radicand.*

Thus,

$$x\sqrt[n]{a}+y\sqrt[n]{a}=(x+y)\sqrt[n]{a}.$$

The expression $(\sqrt{2}+\sqrt{3})$ cannot be simplified. Likewise, $(\sqrt{7}+\sqrt[3]{7})$ cannot be reduced to a more simple form.

To simplify an expression involving the sum of several radicals, first simplify the radicals themselves, and then combine them whenever possible.

Illustration 1.

$$\begin{aligned} &3\sqrt{20x}+\sqrt[3]{32}-\sqrt[3]{\tfrac{27}{2}}+\sqrt[4]{25x^2}+\sqrt{144x^6}\\ &= 3\cdot 2\sqrt{5x}+2\sqrt[3]{4}-3\sqrt[3]{\tfrac{4}{8}}+\sqrt{\sqrt{25x^2}}+12x^3\\ &= 6\sqrt{5x}+2\sqrt[3]{4}-\tfrac{3}{2}\sqrt[3]{4}+\sqrt{5x}+12x^3\\ &= (6+1)\sqrt{5x}+(2-\tfrac{3}{2})\sqrt[3]{4}+12x^3\\ &= 7\sqrt{5x}+\tfrac{1}{2}\sqrt[3]{4}+12x^3. \end{aligned}$$

Exercise 30

Simplify.

1. $7\sqrt{x}+2\sqrt{x}.$
2. $4\sqrt{7}-\sqrt{7}.$
3. $\sqrt{18}+\sqrt{50}.$
4. $\sqrt{24x^3}+\sqrt{6x^3}.$
5. $\sqrt{48}-\sqrt{12}.$
6. $\sqrt{45}-\sqrt{500}.$
7. $\sqrt{75}-13\sqrt{\tfrac{1}{3}}.$
8. $\sqrt[3]{40}-7\sqrt[3]{\tfrac{1}{25}}.$
9. $\sqrt[3]{16}+8\sqrt[3]{\tfrac{1}{4}}+\sqrt[6]{4}.$
10. $x\sqrt[3]{\dfrac{72}{x^2}}-\dfrac{1}{x}\sqrt[3]{\dfrac{x^4}{3}}-\sqrt[3]{9x}.$
11. $\sqrt[3]{\dfrac{a^3}{5}}+\sqrt[3]{\dfrac{b^3}{5}}.$
12. $\sqrt{x^2}+\sqrt{x^3}-\sqrt{4x^3}.$
13. $\sqrt{25x^2+25}-\sqrt{16x^2+16}-\sqrt{x^2+1}.$
14. $\sqrt{75}-\sqrt{98}-\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{4}{3}}.$
15. $\sqrt[5]{a^7b^{11}c^3}+\sqrt[5]{a^2b^6c^{13}}+\sqrt[5]{a^{12}bc^8}.$
16. $\sqrt[3]{(a+b)^4}-\sqrt[3]{a^4+a^3b}-\sqrt[3]{ab^3+b^4}.$
17. $\sqrt{24}-\sqrt{\tfrac{25}{6}}+\sqrt[3]{32}+\sqrt[3]{\tfrac{27}{2}}-\sqrt[4]{36}.$
18. $\sqrt{32}-\sqrt[3]{\dfrac{27x^4}{8}}-\sqrt{\dfrac{9}{8}}-x^2\sqrt[3]{\dfrac{64}{x^2}}+\sqrt[4]{4}.$

19. $\sqrt{80}+\sqrt{125}+\sqrt[9]{125}+\sqrt[3]{135}$.

20. $\sqrt{\dfrac{49}{a^2-1}}-5\sqrt{\dfrac{a+1}{a-1}}+\sqrt{\dfrac{4a-4}{a+1}}$.

55. Multiplication of radicals. Two radicals of the same * order may be multiplied by use of property 1:

$$\sqrt[n]{a}\,\sqrt[n]{b}=\sqrt[n]{ab}.$$

Illustration 1. $\sqrt{3}\,\sqrt{6}=\sqrt{18}=3\sqrt{2}$.

Illustration 2.

$$\begin{aligned}(8+\sqrt{11})(5-3\sqrt{11}) &= 8\cdot 5-8\cdot 3\sqrt{11}+5\sqrt{11}-3\sqrt{11}\,\sqrt{11}\\ &= 40-24\sqrt{11}+5\sqrt{11}-33=7-19\sqrt{11}\end{aligned}$$

Exercise 31

Perform the indicated multiplications and simplify.

1. $\sqrt{5}\,\sqrt{10}$.
2. $5\sqrt{2x}\,\sqrt{22x}$.
3. $\sqrt[3]{14x^2}\,\sqrt[3]{4x^2}$.
4. $\sqrt[3]{4}\,\sqrt[3]{6}$.
5. $\sqrt[3]{\frac{1}{4}}\,\sqrt[3]{\frac{1}{6}}$.
6. $\sqrt{\frac{2}{5}}\,\sqrt{2}$.
7. $(5\sqrt{ab})^3$.
8. $(10\sqrt{3})^4$.
9. $(7a^7\sqrt[3]{x^2y})^2$.
10. $(2a^2\sqrt{10x})^3$.
11. $\sqrt[4]{ab^2c^3}\,\sqrt[4]{ab^3c^3}\,\sqrt[4]{a^3bc}$.
12. $(\sqrt[3]{7a^2b})^4$.
13. $(\sqrt{x}-\sqrt{y})^2$.
14. $(4-3\sqrt{7})^2$.
15. $(5\sqrt{2}+\sqrt{6})^2$.
16. $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})$.
17. $(3-\sqrt{7})(3+\sqrt{7})$.
18. $(3-\sqrt{5})(4-\sqrt{5})$.
19. $(\sqrt{11}-\sqrt{7})(\sqrt{5}-\sqrt{3})$.
20. $\sqrt{3}(\sqrt{2}+\sqrt{3}+\sqrt{6})$.
21. $(5+\sqrt{3})(8-\sqrt{2})$.
22. $(7+\sqrt{3})(1+5\sqrt{3})$.
23. $(3\sqrt{7}+\sqrt{5})^2+(\sqrt{7}-3\sqrt{5})^2$.
24. $(\sqrt[3]{a}-\sqrt[3]{b})(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2})$.
25. $(x-7+\sqrt{2})(x-7-\sqrt{2})$.
26. $\left(x+\dfrac{3+\sqrt{5}}{2}\right)\left(x+\dfrac{3-\sqrt{5}}{2}\right)$.
27. Find the value of $5x^2-13x-1$ if $x=1-\sqrt{2}$.

* See Art. 57 for the case in which their orders are different.

28. Find the value of $x^2 - 5x + 3$ if $x = \dfrac{5 - \sqrt{13}}{2}$.

29. Show by use of substitution that $\dfrac{3 + \sqrt{5}}{4}$ is a root of the equation $4x^2 - 6x + 1 = 0$.

30. Show by use of substitution that $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ is a root of the equation $ax^2 + bx + c = 0$.

56. Division of radicals, rationalizing denominators. Two radicals of the same * order may be divided by use of property 2:

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

Illustration 1. $\dfrac{\sqrt[3]{10x^4}}{\sqrt[3]{2x}} = \sqrt[3]{\dfrac{10x^4}{2x}} = \sqrt[3]{5x^3} = x\sqrt[3]{5}.$

Illustration 2. $\dfrac{\sqrt{5}}{\sqrt{3}} = \sqrt{\dfrac{5}{3}} = \sqrt{\dfrac{5}{3} \cdot \dfrac{3}{3}} = \dfrac{\sqrt{15}}{3}.$

If a radical occurs in the denominator of a fraction, we can rationalize the denominator (free it of radicals) by multiplying the numerator and denominator by a suitable quantity.

Illustration 3. $\dfrac{\sqrt{5}}{\sqrt{3}} = \dfrac{\sqrt{5}}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\sqrt{15}}{3}.$

Illustration 4. $\dfrac{\sqrt[3]{7}}{\sqrt[3]{2}} = \dfrac{\sqrt[3]{7}}{\sqrt[3]{2}} \dfrac{\sqrt[3]{4}}{\sqrt[3]{4}} = \dfrac{\sqrt[3]{28}}{2}.$

Illustration 5. $\dfrac{2}{\sqrt{6}} = \dfrac{2}{\sqrt{6}} \cdot \dfrac{\sqrt{6}}{\sqrt{6}} = \dfrac{2\sqrt{6}}{6} = \dfrac{\sqrt{6}}{3}.$

In obtaining a decimal approximation for an expression involving radicals, it is desirable to have the denominator free of radicals before performing the computation. Notice, in Illustration 5, the labor involved in computing $\dfrac{2}{\sqrt{6}} = \dfrac{2}{2.449}$, as in contrast to the simplicity of computing $\dfrac{\sqrt{6}}{3} = \dfrac{2.449}{3} = .816.$

* See Art. 57 for the case in which their orders are different.

If the denominator of a fraction is a binomial in which one or both terms are of the form $a\sqrt{b}$, multiply the numerator and denominator by the *denominator with the sign of the second term changed.* This rationalizes the denominator.

Illustration 6.

$$\frac{5\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}} = \frac{5\sqrt{7}+\sqrt{3}}{\sqrt{7}-\sqrt{3}}\cdot\frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}}$$

$$= \frac{5\sqrt{7}\,\sqrt{7}+5\sqrt{7}\,\sqrt{3}+\sqrt{3}\,\sqrt{7}+\sqrt{3}\,\sqrt{3}}{(\sqrt{7})^2-(\sqrt{3^2})}$$

$$= \frac{35+5\sqrt{21}+\sqrt{21}+3}{7-3} = \frac{38+6\sqrt{21}}{4}$$

$$= \frac{19+3\sqrt{21}}{2}.$$

Exercise 32

Perform the indicated divisions. Rationalize all denominators.

1. $\dfrac{\sqrt{10}}{\sqrt{2}}$.
2. $\dfrac{\sqrt{22}}{\sqrt{11}}$.
3. $\dfrac{\sqrt{7}}{\sqrt{14}}$.
4. $\dfrac{\sqrt{6}}{\sqrt{5}}$.
5. $\dfrac{6}{\sqrt{3}}$.
6. $\dfrac{10}{\sqrt{5}}$.
7. $\dfrac{\sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{6}}}$.
8. $\dfrac{4}{\sqrt[3]{2}}$.
9. $\dfrac{\sqrt[4]{72x^3}}{\sqrt[4]{8x^{11}}}$.
10. $\dfrac{\sqrt[3]{-21}}{\sqrt[3]{14}}$.
11. $\dfrac{\sqrt[3]{77x^5}}{\sqrt[3]{11x}}$.
12. $\dfrac{\sqrt{\frac{1}{3}xy}}{\sqrt{2yz}}$.
13. $\dfrac{\sqrt[3]{24}}{\sqrt[3]{15}}$.
14. $\dfrac{\sqrt[5]{a^4x^2}}{\sqrt[5]{ax^3}}$.
15. $\dfrac{1}{2\sqrt{3}+\sqrt{7}}$.
16. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.
17. $\dfrac{\sqrt{7}+3\sqrt{5}}{\sqrt{7}-\sqrt{5}}$.
18. $\dfrac{4+\sqrt{3}}{2+\sqrt{3}}$.
19. $\dfrac{\sqrt{6}+\sqrt{5}}{\sqrt{2}}$.
20. $\dfrac{6-2\sqrt{5}}{3-\sqrt{5}}$.
21. $\dfrac{\sqrt{6}-2\sqrt{3}}{4\sqrt{3}-5\sqrt{2}}$.

*22. $\dfrac{1}{\sqrt{3}+\sqrt{2}-1}$.

* *Hint:* Multiply top and bottom by $\sqrt{3}+\sqrt{2}+1$.

Use Table I to compute to three decimal places the value of each of the following expressions before and after rationalizing the denominator.

23. $\dfrac{1}{\sqrt{2}}$. **24.** $\dfrac{1}{\sqrt[3]{5}}$. **25.** $\dfrac{14}{\sqrt[3]{7}}$.

26. $\dfrac{1}{2\sqrt{10}}$. **27.** $\dfrac{1}{5+2\sqrt{7}}$. **28.** $\dfrac{1}{2\sqrt{3}-\sqrt{2}}$.

57. Changing the order of a radical. Two or more radicals having different orders may be reduced to radicals having the same order. This is accomplished by the use of fractional exponents. The common order of the radicals is the L.C.M. of their original orders.

Example 1. Arrange according to size: $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[6]{130}$.

Solution. The L.C.M. of the orders 2, 3, and 6 is 6.

$$\sqrt{5} = 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3} = \sqrt[6]{125}.$$
$$\sqrt[3]{11} = 11^{\frac{1}{3}} = 11^{\frac{2}{6}} = \sqrt[6]{11^2} = \sqrt[6]{121}.$$

Since $121 < 125 < 130$, it follows that

$$\sqrt[3]{11} < \sqrt{5} < \sqrt[6]{130}.$$

To multiply or divide two radicals with different orders, first change them to the same order, then apply property 1 or property 2.

Illustration 1.

$$\sqrt[4]{2x}\,\sqrt[6]{x^5} = (2x)^{\frac{1}{4}}x^{\frac{5}{6}} = (2x)^{\frac{3}{12}}x^{\frac{10}{12}} = \sqrt[12]{(2x)^3}\,\sqrt[12]{x^{10}} = \sqrt[12]{8x^{13}} = x\sqrt[12]{8x}.$$

Illustration 2. $$\frac{\sqrt{a}}{\sqrt[3]{a^2}} = \frac{a^{\frac{1}{2}}}{a^{\frac{2}{3}}} = \frac{a^{\frac{3}{6}}}{a^{\frac{4}{6}}} = \frac{\sqrt[6]{a^3}}{\sqrt[6]{a^4}} = \sqrt[6]{\frac{a^3}{a^4}\cdot\frac{a^2}{a^2}} = \frac{\sqrt[6]{a^5}}{a}.$$

58. Miscellaneous operations on radicals. A root of a radical may be simplified by use of property 3.

Illustration 1.

$$\sqrt{49\sqrt[3]{100x^{14}}} = \sqrt{49}\,\sqrt{\sqrt[3]{100x^{14}}} = 7\sqrt[3]{\sqrt{100x^{14}}} = 7\sqrt[3]{10x^7}$$
$$= 7x^2\sqrt[3]{10x}.$$

A factor can be introduced under a radical sign.

Illustration 2. $$2a\sqrt[3]{5a} = \sqrt[3]{8a^3}\,\sqrt[3]{5a} = \sqrt[3]{40a^4}.$$

This operation, which does not simplify the radical, can be used in comparing the size of two quantities. For example, $5\sqrt{3}$ is larger than $6\sqrt{2}$ because $5\sqrt{3} = \sqrt{75}$, whereas $6\sqrt{2} = \sqrt{72}$.

Exercise 33

Change both radicals to the same order. State which radical is the larger.

1. $\sqrt{2}, \sqrt[5]{6}$. **2.** $\sqrt{7}, \sqrt[3]{19}$. **3.** $\sqrt[4]{3}, \sqrt[6]{5}$. **4.** $\sqrt[3]{4}, \sqrt[4]{6}$.

Perform the indicated operations and simplify.

5. $\sqrt[3]{a}\sqrt[6]{a}$. **6.** $\sqrt[4]{a}\sqrt[5]{a}$. **7.** $\sqrt[3]{2}\sqrt[4]{2}$. **8.** $\sqrt{10}\sqrt[4]{10}$.

9. $\sqrt{2}\sqrt[3]{x}$. **10.** $\sqrt[4]{5}\sqrt[6]{a^5}$. **11.** $\sqrt[5]{3}\sqrt[10]{2}$. **12.** $\sqrt{5}\sqrt[3]{2}$.

13. $\sqrt[4]{ax^3}\sqrt[6]{ax^5}$. **14.** $\sqrt[3]{a^2b}\sqrt[4]{a^2b^3}$. **15.** $\dfrac{\sqrt[3]{a}}{\sqrt[7]{a^2}}$. **16.** $\dfrac{\sqrt{a}}{\sqrt[5]{a^2}}$.

17. $\dfrac{\sqrt[5]{3}}{\sqrt{x}}$. **18.** $\dfrac{\sqrt[8]{x}}{\sqrt[4]{2}}$. **19.** $\dfrac{\sqrt[3]{9}}{\sqrt{10}}$. **20.** $\dfrac{\sqrt{10}}{\sqrt[3]{2}}$.

Simplify.

21. $\sqrt{\sqrt[3]{25}}$. **22.** $\sqrt{\sqrt{\sqrt[3]{x^2}}}$. **23.** $\sqrt[3]{27\sqrt[4]{8a^3}}$.

24. $\sqrt[4]{81\sqrt[3]{4y^2}}$. **25.** $\sqrt[5]{32\sqrt{x^{15}}}$. **26.** $\sqrt[3]{64\sqrt{125x^9}}$.

Introduce all coefficients under the radical signs.

27. $10x\sqrt{7x}$. **28.** $2a\sqrt[4]{3x}$. **29.** $5a^2\sqrt[3]{2a^2}$.

30. $\dfrac{3x}{2y}\sqrt{2xy}$. **31.** $x\sqrt[5]{3 - x^2}$. **32.** $2x^3\sqrt[4]{2x^2}$.

33. Find the decimal value of $2\sqrt{15}$ after introducing the 2 under the radical.

59. Complex numbers. It has been demonstrated (Art. 47) that a negative number cannot have a real square root. In order to solve certain kinds of equations (such as $x^2 + 1 = 0$), we introduce a new type of number — an imaginary number. For our convenience we shall use the letter i to designate $\sqrt{-1}$. Then i is a number whose square is -1.

Definition. $\quad i = \sqrt{-1}$.

Consequence. $\quad i^2 = -1$.

Square roots of negative numbers can be expressed in terms of the imaginary unit i. Thus $\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$ and $\sqrt{-7} = \sqrt{7}\sqrt{-1} = i\sqrt{7}$. Such numbers as $\sqrt{-1}$, $\sqrt{-9}$, and $\sqrt{-7}$ are called pure imaginary numbers. In general,

A **pure imaginary number** *is a square root of a negative number.*

Illustration 1. The two square roots of -9 are $\pm\sqrt{-9} = \pm 3i$.

In solving the equation $x^2 - 6x + 34 = 0$,* we obtain as one of the roots $x = 3 + 5i$. Notice that if $3 + 5i$ is substituted for x in the equation $x^2 - 6x + 34 = 0$, we get

$$(3 + 5i)^2 - 6(3 + 5i) + 34 = 0$$
$$9 + 30i + 25i^2 - 18 - 30i + 34 = 0.$$

If i^2 is replaced by -1, we obtain

$$43 + 30i - 30i - 43 = 0.$$

This shows that $3 + 5i$ is a perfectly good root of the equation provided we understand that i is a number whose square is -1. A number (such as $3 + 5i$) that is the sum of a real number and a pure imaginary number is called a complex number. In general,

A **complex number** *is a number of the form* $\boldsymbol{a + bi}$, *where a and b are real numbers and* $i = \sqrt{-1}$. If $a = 0$, the complex number $a + bi$ becomes bi, a pure imaginary number. If $b = 0$, the complex number $a + bi$ becomes a, a real number. Hence we see that real numbers and pure imaginary numbers are special cases of complex numbers. If $b \neq 0$, the complex number $a + bi$ is called an **imaginary number.**†

Before performing any algebraic operations on complex numbers, we should write them in the form $a + bi$. Thus, $-7 - \sqrt{-81} = -7 - \sqrt{81}\sqrt{-1} = -7 - 9i$. This procedure is suggested to avoid mistakes such as $\sqrt{-3}\sqrt{-3} = \sqrt{(-3)(-3)} = \sqrt{9} = 3$. This is obviously incorrect because, by the definition of square root, $\sqrt{-3}$

* See Art. 63.

† It is unfortunate that the term "imaginary" was applied to these numbers. In a certain sense, $\sqrt{-1}$ is no more imaginary than -1. A negative number furnishes a convenient way to indicate a debt or a deficit. So-called imaginary numbers are very useful in science, especially in the theory of alternating currents in electricity.

is a number which when multiplied by itself becomes -3. The correct way to handle this is

$$\sqrt{-3}\sqrt{-3} = i\sqrt{3} \cdot i\sqrt{3} = 3i^2 = -3.$$

The result agrees with the definition of square root.

In performing algebraic operations on complex numbers, treat i like any other letter, but replace i^2 (whenever it appears) *with* -1.

Illustration 2.

$$(2+3i)+(5-9i) = (2+5)+(3-9)i = 7-6i.$$

Illustration 3.

$$\begin{aligned}(2+3i)(5-9i) &= 10-3i-27i^2\\ &= 10-3i+27 \text{ (since } i^2=-1)\\ &= 37-3i.\end{aligned}$$

To compute positive integral powers of i, remember that $i^2=-1$. Hence

$$\begin{aligned}i^3 &= i^2\cdot i = -i\\ i^4 &= (i^2)^2 = (-1)^2 = 1\\ i^5 &= i^4\cdot i = i\\ i^6 &= i^4\cdot i^2 = 1(-1) = -1\\ i^{23} &= i^{20}\cdot i^3 = (i^4)^5 i^3 = 1^5(-i) = -i.\end{aligned}$$

Obviously any positive integral power of i is equal to one of the four quantities: $1, -1, i, -i$.

Exercise 34

Express in terms of i.

1. $\sqrt{-36}$. 2. $\sqrt{-4}$. 3. $\sqrt{-\frac{25}{49}}$. 4. $\sqrt{-\frac{1}{9}}$.

5. $\sqrt{-11}$. 6. $\sqrt{-5}$. 7. $\sqrt{-12}$. 8. $\sqrt{-18}$.

State the two square roots of each number.

9. -121. 10. $-\frac{9}{16}$. 11. $-\frac{5}{36}$. 12. -13.

Perform the indicated operations and simplify.

13. $(4+9i)+(-1-7i)$. 14. $(3+i)-(2-5i)$.

15. $(8-5i)^2$. 16. $(8+i)i$.

17. $(7-i)(4+i)$. 18. $(2-6i)(-7+i)$.

19. $(5 + 3i)(5 - 3i)$.

20. $(1 + i)^4$.

21. i^7.

22. i^{17}.

23. i^{26}.

24. i^{12}.

25. Find the value of $x^2 - 10x + 26$ if $x = 5 + i$.

26. Find the value of $x^2 + 12x + 37$ if $x = -6 + i$.

27. Show by substitution that $-3 + 2i$ is a root of the equation $x^2 + 6x + 13 = 0$.

28. Show by substitution that $\frac{1}{2} - 3i$ is a root of the equation $4x^2 - 4x + 37 = 0$.

29. Show by substitution that $\frac{2}{3} + i\sqrt{5}$ is a root of $9x^2 - 12x + 49 = 0$.

30. Show by substitution that $-1 - i\sqrt{3}$ is a root of $x^3 = 8$.

Express each of the following complex numbers in the form $a + bi$ and state the values of a and b. Identify the number as real or imaginary. If the number is real, classify it as rational or irrational. If the number is a pure imaginary, identify it as such.

31. $\frac{1}{7} - \sqrt{-16}$.

32. 1947.

33. -44.

34. $\sqrt{-28}$.

35. $1 + \sqrt{2}$.

36. $-\frac{2}{3} + \sqrt{-100}$.

37. $\sqrt{-50}$.

38. $5 - \sqrt[3]{-8}$.

39. 0.

40. $-i\sqrt{-2}$.

Exercise 35

(Miscellaneous problems on exponents and radicals.)

Simplify.

1. $(\frac{9}{16})^{\frac{3}{2}}$.

2. $\dfrac{64^{\frac{1}{2}}}{4}$.

3. $-8^{\frac{4}{3}}$.

4. $(-125)^{\frac{2}{3}}$.

5. $(100x^{100})^{\frac{1}{2}}$.

6. $(27x^{27})^{\frac{4}{3}}$.

7. $36^{-\frac{3}{2}}$.

8. $100^{-\frac{7}{2}}$.

9. $(\frac{1}{8})^{-\frac{2}{3}}$.

10. $(\frac{1}{25})^{-\frac{1}{2}}$.

11. $\left(\dfrac{x^{12}}{16}\right)^{-\frac{3}{4}}$.

12. $\left(\dfrac{243}{x^{20}}\right)^{-\frac{3}{5}}$.

13. $28 - 3 \cdot 4^{\frac{3}{2}}$.

14. $10 \cdot 64^{\frac{5}{6}}$.

15. $6^0 + 48 \cdot 2^{-3}$.

16. $28 - 18 \cdot 3^{-2}$.

17. $1^{-\frac{5}{6}} - 8^{-\frac{5}{3}}$.

18. $0^{\frac{5}{7}} - 4^{\frac{5}{2}}$.

19. $(\frac{9}{4})^{-2}$.

20. $\left(\dfrac{16}{x^{16}}\right)^{\frac{5}{4}}$.

21. $3^{-2} + 4^{-2}$.

22. $16 - 12(-\frac{7}{12})^0$.

23. $\dfrac{5x^{-1}}{(7x)^{-1} + 9y^{-3}}$.

24. $\dfrac{(a + b)^{-2}}{a^{-2} + b^{-2}}$.

25. $\dfrac{1 - 25x^{-4}}{5^{-1} + x^{-2}}$.

26. $\dfrac{1}{x^0 + 9x^{-1} + x^{-2}}$.

27. $\sqrt{\frac{7}{27}}$.

28. $\sqrt{\dfrac{3}{5x^7}}$.

29. $\dfrac{2}{\sqrt[4]{8}}$.

30. $\dfrac{3}{\sqrt[3]{6}}$.

31. $\dfrac{\sqrt{10}}{\sqrt{60}}$.

32. $\dfrac{\sqrt{2x}}{\sqrt{18y}}$.

33. $\dfrac{\sqrt[3]{44xy^2}}{\sqrt[3]{2x^2y}}$.

34. $\dfrac{\sqrt[4]{2ab^{-1}}}{\sqrt[4]{32a^9b^{-5}}}$.

35. $\sqrt{3xy}\,\sqrt{21x}$.

36. $\sqrt[3]{4x^2y}\,\sqrt[3]{6x^2y^8}$.

37. $\dfrac{1}{5}\sqrt{\dfrac{8}{9}} + \dfrac{2}{\sqrt{72}}$.

38. $\sqrt{20} - \sqrt{\frac{49}{5}}$.

39. $\sqrt[3]{88x^8} - x^3\sqrt[3]{\dfrac{11}{x}}$.

40. $x^2\sqrt{\dfrac{75}{x}} + \sqrt[6]{27x^9}$.

41. $\dfrac{8 - \sqrt{2}}{4 - \sqrt{2}}$.

42. $\dfrac{6 - \sqrt[4]{49}}{5 + \sqrt[4]{49}}$.

43. $(2\sqrt{7} - 3\sqrt{11})^2$.

44. $(\sqrt{2} + \sqrt{6})(7 + \sqrt{3})$.

45. $\sqrt[3]{\dfrac{32x^{19}}{y^{48}}}$.

46. $\sqrt{\dfrac{52x^{15}}{y^{16}}}$.

47. $\sqrt[5]{\dfrac{128x^{32}}{81}}$.

48. $\sqrt[4]{\dfrac{32x^5y^2}{x^3y^9}}$.

49. $\sqrt[12]{4x^{40}}$.

50. $\sqrt[10]{9x^{18}}$.

51. $(2r^2\sqrt[4]{st^3})^3$.

52. $(5a^5\sqrt{x^2y^3})^4$.

53. $\sqrt{36x^{-36} + 9}$.

54. $\sqrt[3]{8 + 64x^{-6}}$.

55. $\sqrt[6]{2}\,\sqrt[4]{x^3}$.

56. $\dfrac{\sqrt[3]{x}}{\sqrt{y}}$.

57. $\sqrt[6]{x^{12}\sqrt[3]{4y^2}}$.

58. $\sqrt{(e^x - e^{-x})^2 + 4e^0}$.

59. Reduce $\dfrac{1}{2\sqrt{x^2 + 1}} + \dfrac{1}{2}\left[\sqrt{x^2 + 1} + \dfrac{x^2}{\sqrt{x^2 + 1}}\right]$ to $\sqrt{x^2 + 1}$.

60. Reduce $\dfrac{(1 - x^2)^{\frac{1}{2}} + x^2(1 - x^2)^{-\frac{1}{2}}}{1 - x^2}$ to $\dfrac{1}{(1 - x^2)^{\frac{3}{2}}}$.

chapter 8

Quadratic equations

60. Quadratic equations. A **quadratic** (or second-degree) **equation in x** is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, c are expressions that do not involve x, and $a \neq 0$. The following are examples of quadratic equations in x:

$$7x^2 - 8x - 9 = 0. \qquad 3x^2 + 5x = 0. \qquad 4x^2 = 11. \qquad 6x^2 = 0.$$

What are the values of a, b, c in each equation?

61. First-degree term missing. To solve a quadratic equation in which the first-degree term is missing, first solve for x^2, then extract square roots.

Example 1. Solve for x: $4x^2 - 7 = 0$.

Solution.

$$4x^2 = 7$$
$$x^2 = \tfrac{7}{4}$$
$$x = \pm\sqrt{\frac{7}{4}} = \pm\frac{\sqrt{7}}{2}.$$

The double sign $\pm$ (read "plus or minus") indicates that the two roots of the equation are $+\dfrac{\sqrt{7}}{2}$ and $-\dfrac{\sqrt{7}}{2}$.

Example 2. Solve for x: $3x^2 + 5 = 0$.

Solution.

$$3x^2 = -5.$$

$$x^2 = -\tfrac{5}{3}.$$

$$x = \pm\sqrt{-\frac{5}{3}} = \pm\, i\sqrt{\frac{5}{3}\cdot\frac{3}{3}} = \pm\,\frac{i\sqrt{15}}{3}.$$

The roots of this equation are the pure imaginary numbers $\dfrac{i\sqrt{15}}{3}$ and $\dfrac{-i\sqrt{15}}{3}$.

62. Solution by factoring. The following principle is useful in solving certain kinds of equations.

*If the product of two or more quantities is zero, then at least one of the factors must be zero.** Thus if $uv = 0$, we must conclude that either $u = 0$ or $v = 0$.

To solve a quadratic equation by factoring.

1. *Make one side of the equation equal to zero by transposing terms.*
2. *Factor the other side of the equation* (if possible).
3. *Set each factor equal to zero and solve the resulting linear equations.*

Example 1. Solve for x: $36x^2 = 60x + 24$.

Solution.

$$36x^2 - 60x - 24 = 0.$$

$$3x^2 - 5x - 2 = 0.$$

$$(3x + 1)(x - 2) = 0.$$

$$3x + 1 = 0 \quad \text{or} \quad x - 2 = 0.$$

The roots are $x = -\frac{1}{3}$ and $x = 2$.

Check for $x = -\frac{1}{3}$:

$$36(-\tfrac{1}{3})^2 = 60(-\tfrac{1}{3}) + 24.$$

$$36(\tfrac{1}{9}) = -20 + 24.$$

$$4 = 4. \qquad \text{True.}$$

Check for $x = 2$:

$$36(2)^2 = 60(2) + 24.$$

$$36(4) = 120 + 24.$$

$$144 = 144. \qquad \text{True.}$$

* Note that the product must be *zero*. If $uv = 6$, no conclusion can be drawn as to the values of u or v independently. For example, u can be 3 or 6 or 10, in which case v is 2 or 1 or $\frac{3}{5}$ respectively.

Note also that it is the *product* of the quantities that must be zero. If the *sum* of two quantities is zero, it does not necessarily follow that one of these quantities is zero.

Example 2. Solve for x: $3x^2 = 2x$.

Solution.
$$3x^2 - 2x = 0.$$
$$x(3x - 2) = 0.$$
$$x = 0 \quad \text{or} \quad 3x - 2 = 0.$$

The roots are $x = 0$ and $x = \frac{2}{3}$. The student should check the roots.

Incorrect solution. $3x^2 = 2x$. Divide both sides by x: $3x = 2$, $x = \frac{2}{3}$. In the incorrect solution, the root $x = 0$ was lost. See Art. 28, Ill. 4.

If both factors give the same root, the equation is said to have a double root, i.e., it has two equal roots. Thus the equation $x^2 - 10x + 25 = 0$ has 5 as a double root.

Exercise 36

Solve for x and check if directed by the instructor.

1. $5x^2 = 55$.
2. $7x^2 = 14$.
3. $3x^2 + 48 = 0$.
4. $4x^2 + 12 = 0$.
5. $x^2 + 15 = 3 - x^2$.
6. $2x^2 - 5 = 6x^2 + 4$.
7. $2x^2 + 11 = 5x^2 + 7$.
8. $4x^2 = 3 - x^2$.
9. $\dfrac{x}{4} - \dfrac{1}{x} = \dfrac{7x}{10} - \dfrac{6}{5x}$.
10. $\dfrac{5x}{6} + \dfrac{1}{4x} = \dfrac{1}{x} - \dfrac{x}{2}$.
11. $16r^2x^2 + s^2 = r^2$.
12. $a^2x^2 - a = x^2 + 1$.

Solve for x by factoring. Check all roots.

13. $x^2 - 5x - 6 = 0$.
14. $x^2 - 5x + 6 = 0$.
15. $x^2 + 11x + 28 = 0$.
16. $x^2 + 2x - 15 = 0$.
17. $40x^2 - 48x + 8 = 0$.
18. $18x^2 + 42x + 12 = 0$.
19. $x(7x + 5) = 2$.
20. $4x(x - 1) + 1 = 0$.
21. $9x^2 + 60x + 100 = 0$.
22. $5x^2 = 32x + 21$.
23. $3x^2 + 5 = 16x$.
24. $-11x^2 = 18x + 7$.
25. $4x^2 = 5x$.
26. $9x^2 = 6x$.
27. $8x^2 + 3x = 0$.
28. $2x - 7x^2 = 0$.
29. $\dfrac{x + 2}{x^2 - 5x + 4} + \dfrac{1}{x - 1} + \dfrac{1}{3} = 0$.
30. $\dfrac{2x + 5}{x^2 + 6x + 9} + \dfrac{3}{x^2 + 3x} - \dfrac{1}{x} = 0$.
31. $x^2 + 18bx + 45b^2 = 0$.
32. $48a^2x^2 - 8abx - b^2 = 0$.

63. Solution by completing the square. A quadratic equation can be solved by factoring, only if one side is factorable when the other side is zero. A quadratic equation can always be solved by a process called **completing the square.**

The expression $x^2 + qx$ will become a perfect square if we add $\frac{q^2}{4}$. This gives $x^2 + qx + \frac{q^2}{4}$ which is the square of $\left(x + \frac{q}{2}\right)$. The term that was added is $\left(\frac{q}{2}\right)^2$ which is *the square of half the coefficient of x.*

Illustration 1. To complete the square of $x^2 - \frac{6}{5}x$, add $[\frac{1}{2}(-\frac{6}{5})]^2 = [-\frac{3}{5}]^2 = \frac{9}{25}$. This gives $x^2 - \frac{6}{5}x + \frac{9}{25}$ or $(x - \frac{3}{5})^2$.

Note that this method applies only when the coefficient of x^2 is 1.

To solve a quadratic equation in x by completing the square.

1. *Place all terms involving x on the left side. Place all other terms on the right side.*
2. *Divide through by the coefficient of x^2.*
3. *Add to both sides the square of one-half the coefficient of x. This completes the square on the left side.*
4. *Write the left side as a perfect square.*
5. *Take the square root of both sides, using the double sign on the right side.**

Example 1. Solve for x: $8x^2 + 6x - 5 = 0$.

Solution.
$$8x^2 + 6x = 5.$$
$$x^2 + \tfrac{3}{4}x = \tfrac{5}{8}.$$

The coefficient of x is $\frac{3}{4}$. Half of $\frac{3}{4}$ is $\frac{3}{8}$. Add $(\frac{3}{8})^2$ to both sides.

$$x^2 + \tfrac{3}{4}x + (\tfrac{3}{8})^2 = \tfrac{5}{8} + \tfrac{9}{64}.$$
$$(x + \tfrac{3}{8})^2 = \tfrac{49}{64}.$$
$$x + \tfrac{3}{8} = \pm\tfrac{7}{8}.$$

Hence
$$x = -\tfrac{3}{8} + \tfrac{7}{8} = \tfrac{4}{8} = \tfrac{1}{2}.$$
$$x = -\tfrac{3}{8} - \tfrac{7}{8} = -\tfrac{10}{8} = -\tfrac{5}{4}.$$

The student should check the roots $\frac{1}{2}$ and $-\frac{5}{4}$.

* If the double sign is used on both sides, the four resulting equations are equivalent to the two equations that develop from Step 5.

Example 2. Solve for x: $5x^2 + 2 = 8x$.

Solution.

$$5x^2 - 8x = -2.$$

$$x^2 - \tfrac{8}{5}x = -\tfrac{2}{5}.$$

Half of $-\frac{8}{5}$ is $-\frac{4}{5}$. Add $(-\frac{4}{5})^2 = \frac{16}{25}$ to both sides.

$$x^2 - \tfrac{8}{5}x + \tfrac{16}{25} = -\tfrac{2}{5} + \tfrac{16}{25}.$$

$$(x - \tfrac{4}{5})^2 = \tfrac{6}{25}.$$

$$x - \frac{4}{5} = \pm\sqrt{\frac{6}{25}} = \pm\frac{\sqrt{6}}{5}.$$

$$x = \frac{4}{5} \pm \frac{\sqrt{6}}{5} = \frac{4 \pm \sqrt{6}}{5}.$$

The exact roots are $\dfrac{4 + \sqrt{6}}{5}$ and $\dfrac{4 - \sqrt{6}}{5}$.

The approximate roots, correct to three decimal places, are $\dfrac{4 + 2.449}{5} = 1.290$ and $\dfrac{4 - 2.449}{5} = 0.310$. The student should check one of the exact roots.

Exercise 37

Solve for x by completing the square. Check as directed by the instructor. Use Table I to approximate irrational roots to three decimal places.

1. $x^2 - 10x + 22 = 0.$
2. $x^2 + 6x - 1 = 0.$
3. $x^2 + 2x - 6 = 0.$
4. $x^2 - 12x + 31 = 0.$
5. $x^2 - 8x + 25 = 0.$
6. $x^2 - 4x + 5 = 0.$
7. $x^2 - 12x - 864 = 0.$
8. $x^2 + 8x - 384 = 0.$
9. $9x^2 + 6x - 35 = 0.$
10. $25x^2 - 20x - 12 = 0.$
11. $x^2 - 3x - 1 = 0.$
12. $x^2 + 5x + 2 = 0.$
13. $3x^2 = 4x + 1.$
14. $2x^2 = 4x + 1.$
15. $12x - 9x^2 = 104.$
16. $4x - 4x^2 = 101.$
17. $2x^2 - 5x + 1 = 0.$
18. $5x^2 + 3x - 4 = 0.$
19. $4x^2 + 12x - 5 = 0.$
20. $6x^2 - 3x - 2 = 0.$

21. Given the equation $x^2 - 2x - y^2 - 6 = 0$. (*a*) Solve for y in terms of x. (*b*) Solve for x in terms of y.

22. Given the equation $y^2 - 6y + x^2 + 5 = 0$. (*a*) Solve for y in terms of x. (*b*) Solve for x in terms of y.

23. Solve for t: $s = v_0t + \frac{1}{2}gt^2$.

64. Solution by the quadratic formula. We shall now solve the general equation $ax^2 + bx + c = 0$ by completing the square. This will provide us with a formula which may be used to solve any quadratic equation.

$$ax^2 + bx + c = 0.$$
$$ax^2 + bx = -c.$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}.$$
$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$
$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}.$$
$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The roots of the general quadratic equation

$$\boldsymbol{ax^2 + bx + c = 0}$$

are

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}.$$

This is the quadratic formula. It should be memorized.

To solve a quadratic equation by use of the quadratic formula, write the equation in the standard form $ax^2 + bx + c = 0$, list the values of a, b, c, and then substitute in the formula.

Example 1. Solve for x: $3x^2 = 8x + 5$.

Solution. $$3x^2 - 8x - 5 = 0.$$

In this case, $a = 3$, $b = -8$, $c = -5$. The formula gives us

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-5)}}{6}$$
$$= \frac{8 \pm \sqrt{124}}{6} = \frac{8 \pm 2\sqrt{31}}{6} = \frac{4 \pm \sqrt{31}}{3}.$$

Exercise 38

Solve for x by using the quadratic formula. Check as directed by the instructor.

1. $x^2 + 3x + 1 = 0.$
2. $x^2 - 7x + 4 = 0.$
3. $x^2 - 30x + 216 = 0.$
4. $x^2 + 24x + 119 = 0.$
5. $x^2 - 10x + 34 = 0.$
6. $x^2 + 8x + 41 = 0.$
7. $8x^2 + 2x = 3.$
8. $10x^2 = 11x - 3.$
9. $2x^2 = 5x + 1.$
10. $3x^2 = 5x + 1.$
11. $x^2 - x + 1 = 0.$
12. $x^2 - 7x + 13 = 0.$
13. $4x^2 - 4x - 1 = 0.$
14. $2x^2 - 6x - 1 = 0.$
15. $5x^2 - 4x - 2 = 0.$
16. $2x^2 - 4x - 3 = 0.$
17. $3x^2 + 2x - 2 = 0.$
18. $3x^2 + 4x - 6 = 0.$
19. $3x^2 + 7hx + h^2 - 1 = 0.$
20. $5h^2x^2 - 6hx - 2 = 0.$
21. Solve for y in terms of x: $2x^2 + 5xy + y^2 + 4x - y = 7.$
22. Solve for x in terms of y: $x^2 + 3xy + y^2 - x + 2y = 4.$

65. Solution of quadratic equations. If a quadratic equation contains no first-degree term, it should be solved by extracting square roots (Art. 61). A quadratic equation should be solved by factoring only if the factors can be easily recognized. All other quadratic equations should be solved by completing the square or by use of the quadratic formula. The method of completing the square is usually quicker in case the coefficient of x^2 is 1, especially if the coefficient of x is an even number. For example, the equation $x^2 - 4x - 396 = 0$ can be solved more quickly by completing the square than by the quadratic formula. This equation should not be solved by factoring because the factors are not easily recognized.

66. Stated problems. In solving a stated problem we sometimes encounter a quadratic equation. The two roots of this equation should be checked against the conditions of the original problem rather than in the resulting equation. Sometimes one or both solutions must be rejected because they lack interpretation in the light of the given problem.

The student should review Art. 30 before proceeding.

Example 1. A ship leaves port at noon and sails east at 10 miles per hour. Another ship leaves the same port at 1 P.M. and sails north at 20 miles per hour. When are the ships 50 miles apart?

Solution.

Let t = number of hours that elapse between 1 P.M. and the instant when the ships are 50 miles apart.

Then $10 + 10t$ = number of miles first ship is east of port.

And $20t$ = number of miles second ship is north of port.

The Pythagorean theorem gives us

$$(10 + 10t)^2 + (20t)^2 = 50^2,$$

which reduces to

$$5t^2 + 2t - 24 = 0.$$

The roots of this equation are

$$t = 2 \quad \text{and} \quad t = -\tfrac{12}{5} = -2\tfrac{2}{5}.$$

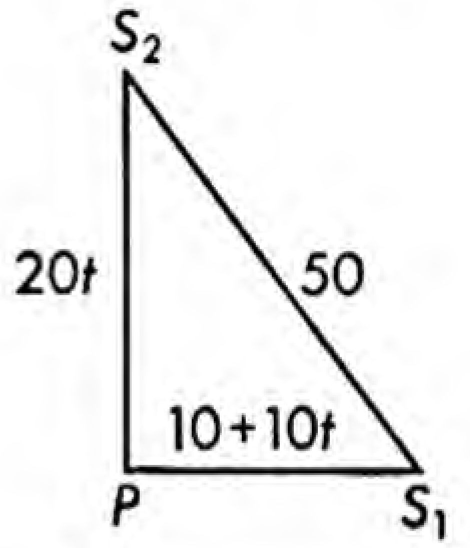


FIG. 11

The solution $t = 2$ is acceptable because it satisfies the conditions of the problem. The solution $t = -2\frac{2}{5}$ must be rejected because the position of the second boat is not defined for negative values of t.

We conclude that the ships are 50 miles apart at 3 P.M.

Exercise 39

Solve by the most convenient method.

1. $x^2 - 16 = 0.$

2. $x^2 + 16 = 0.$

3. $x^2 - 16x = 0.$

4. $7x^2 = 0.$

5. $10x^2 + 50x - 140 = 0.$

6. $12x^2 - 36x - 120 = 0.$

7. $x^2 - 14x + 65 = 0.$

8. $2x^2 + 6x - 3 = 0.$

9. $3x^2 - 4x - 8 = 0.$

10. $4x^2 - 4x - 7 = 0.$

11. $x^2 + 1.6x - .36 = 0.$

12. $.5x^2 - .7x - .255 = 0.$

13. Solve for y: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$

14. Solve $x^2 + y^2 = r^2$, (*a*) for y, (*b*) for r.

15. Solve for t: $s = \frac{1}{2}gt^2.$

16. Solve for r: $V = \pi r^2 h.$

Solve for x and check all roots.

17. $\dfrac{15}{x^2 - x - 6} - \dfrac{x}{x - 3} - \dfrac{1}{2} = 0.$

18. $\dfrac{7}{x^2 - 5x + 6} + \dfrac{3}{x - 2} + \dfrac{1}{3} = 0.$

19. $\dfrac{7}{x^2 - 3x + 2} + \dfrac{5}{x - 1} + \dfrac{3}{2} = 0.$

20. $\dfrac{11}{4x - 8} + \dfrac{5x}{4x + 8} - \dfrac{17}{2x^2 - 8} - \dfrac{3}{4} = 0.$

21. $\dfrac{x + 4}{x^2 - x - 2} + \dfrac{x + 2}{x + 1} - \dfrac{1}{5} = 0.$

22. $\dfrac{4x^2 - 1}{x^2 - 6x + 8} - \dfrac{9}{x - 4} - 2 = 0.$

23. A rectangle has an area of 288 square feet. Find its dimensions if the length exceeds the width by 2 feet.

24. One leg of a right triangle is 4 feet less than the hypotenuse. The other leg is 8 feet less than the hypotenuse. How long is the hypotenuse?

25. A cement sidewalk is to be laid around a rectangular garden that is 20 feet long and 12 feet wide. Find the width of the walk if its area is to be 144 square feet.

26. A rectangular lawn is 40 feet long and 30 feet wide. A boy cuts the grass by starting on the edge and mowing around the perimeter. How wide will the mowed strip be when half of the lawn has been cut?

27. The sum of the squares of three consecutive positive integers is 302. Find the integers.

28. Find two consecutive odd integers whose product is 399.

29. An airplane traveled 48 miles with the wind and then returned to its starting point against the wind. The wind velocity was 20 mph. If the round trip required 1 hour, find the plane's airspeed (speed in still air).

30. A motorboat required 1 hour to travel 8 miles upstream and 6 miles back on a river whose current flows 3 mph. How fast can the motorboat travel in still water?

31. A man traveled 60 miles by auto and then covered the remaining 490 miles of his journey by plane. The plane's speed was 100 mph greater than that of the auto. If the entire trip required 5 hours, find the speed of the auto.

32. A train travels 100 miles in 50 minutes less time than it takes a bus to cover the same distance. Find the speeds of the two vehicles if the train moves 20 mph faster than the bus.

33. A railroad track crosses a highway at right angles. A train is 8 miles from the intersection and approaches it at 2 miles per minute. An auto is 7 miles from the crossing and is leaving it at 1 mile per minute. In how many minutes will the train and auto be 10 miles apart?

34. A ship leaves port at 2 P.M. and sails south at 10 mph. Another ship leaves the same port at 4 P.M. and sails west at 10 mph. When are they 60 miles apart?

35. The number of diagonals of an n-sided polygon is $\frac{1}{2}n(n-3)$. How many sides has a polygon with 65 diagonals?

36. The area of the L-beam in Fig. 12 is 10.25 square inches. Find x.

37. A group of people rented a hall for \$30. When two people decided not to participate, the share of each of the others was increased by \$0.50. How many people were in the original group?

38. A rectangular piece of tin is twice as long as it is wide. From each corner a 3-inch square is cut. The sides and ends are turned up to form an open box with a capacity of 648 cubic inches. Find the dimensions of the piece of tin.

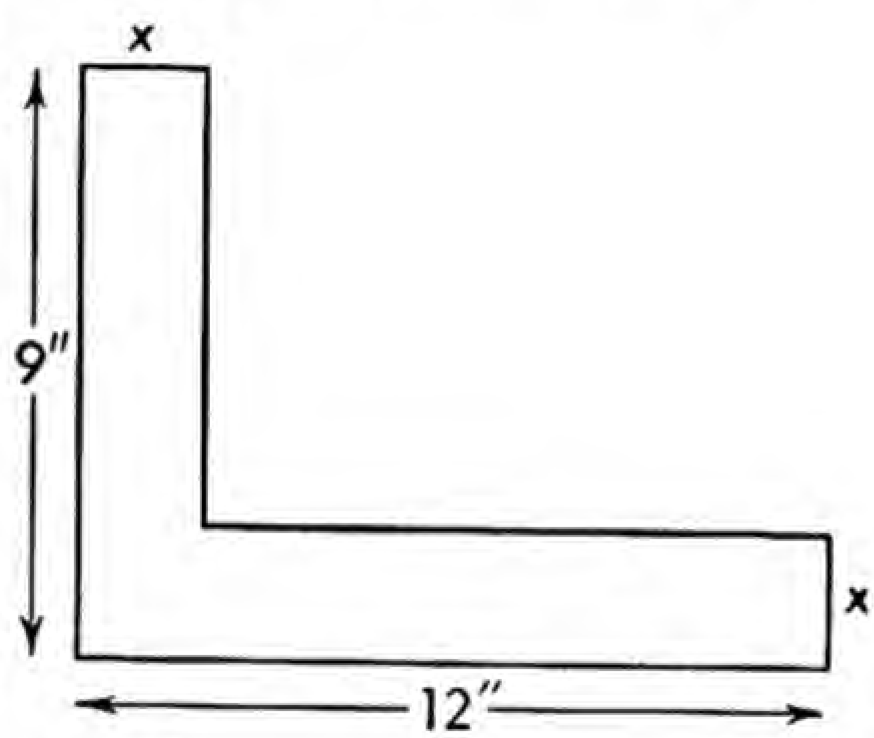


FIG. 12

39. If an object is thrown vertically *upward* with a velocity of v_0, then at the end of t seconds its distance *above* the starting point is

$$s = v_0t - \tfrac{1}{2}gt^2,$$

where s is in feet, v_0 is in feet per second, and $g = 32$ approximately.

A bullet is shot vertically upward from the earth with a velocity of 96 feet per second. In how many seconds will the bullet be 128 feet above the ground? Explain the two solutions.

40. If an object is thrown vertically *downward*, the formula in Prob. 39 becomes

$$s = v_0t + \tfrac{1}{2}gt^2,$$

where s is the distance *below* the starting point.

(*a*) A dive bomber traveling 480 feet per second vertically downward releases a bomb 1024 feet above the earth. How long will it take the bomb to reach the earth? (*b*) How long would it take the bomb to reach the earth if it had been released by a bomber flying in a horizontal plane?

41. A grocer paid \$40 for several bushels of fruit. After discarding 3 bushels that spoiled, he sold the remainder at a profit of 40 cents per bushel. How many bushels did he buy if he gained \$4 on the transaction?

42. An army 3 miles long is marching 4 miles per hour. A messenger leaves the rear and runs to the front of the column. He returns immediately to the rear and arrives there exactly 1 hour after he left. How fast did the messenger travel?

67. Equations involving radicals. An irrational equation is one that contains the unknown in some radicand. In solving an irrational equation that involves square roots, it is usually necessary to square both sides of the equation. This operation may lead to extraneous roots (Art. 28, III). Whenever an equation is squared,* all results should be checked by substitution in the original equation.

Illustration 1. The only root of the equation $x - 3 = 2$ is $x = 5$. If both sides are squared, we get $x^2 - 6x + 9 = 4$, the roots of which are $x = 5$ and $x = 1$. The value $x = 1$ was introduced by squaring.

To solve an irrational equation involving square roots.

1. Isolate the most complicated radical by placing it alone on one side of the equation.

2. Square both sides of the equation. This eliminates the radical that was isolated.

3. Repeat Steps 1 and 2 until all radicals have been eliminated. Solve the resulting equation.

4. Check each value in the original equation.

A similar method applies to radicals of higher order.

Example 1. Solve: $3\sqrt{x} + 4 = x.$

Solution. $3\sqrt{x} = x - 4$

Square: $9x = x^2 - 8x + 16.$

$$x^2 - 17x + 16 = 0; \quad (x - 1)(x - 16) = 0.$$

The possible roots are $x = 1$ and $x = 16.$

Check for $x = 1$: $3\sqrt{1} + 4 = 1$

$7 = 1$ False.

Hence $x = 1$ is an extraneous root.

* Or raised to any positive integral power.

Check for $x = 16$: $3\sqrt{16} + 4 = 16$

$$12 + 4 = 16 \quad \text{True.}$$

Conclusion. The only root of the given equation is $x = 16$.

It should be remembered (Art. 48) that $\sqrt{a}$ or $a^{\frac{1}{2}}$ means the *positive* square root.

Example 2. Solve: $\sqrt{4x+1} + \sqrt{x-2} = 3$.

Solution. $\sqrt{4x+1} = 3 - \sqrt{x-2}$

Square: $(\sqrt{4x+1})^2 = (3 - \sqrt{x-2})^2$

$$4x + 1 = 9 - 6\sqrt{x-2} + x - 2$$

Isolate radical: $3x - 6 = -6\sqrt{x-2}$

Divide by 3: $x - 2 = -2\sqrt{x-2}$

Square: $x^2 - 4x + 4 = 4x - 8$

$$x^2 - 8x + 12 = 0; \quad (x-6)(x-2) = 0.$$

The student should check both possible roots and show that $x = 2$ is the only solution.

Example 3. Solve: $\sqrt{3x+3} = \sqrt{x+3} + \sqrt{x}$.

Partial solution.

Square: $3x + 3 = x + 3 + 2\sqrt{(x+3)x} + x$

Isolate radical: $x = 2\sqrt{x^2 + 3x}$

Square: $x^2 = 4x^2 + 12x$.

The student should finish the problem and show that the roots are $x = 0$ and $x = -4$.

Sometimes we can tell by inspection that an irrational equation has no root. For example, the equation $\sqrt{5x+7} = -2$ cannot have a solution because the left side must be positive or zero by our definition of $\sqrt{a}$ (Art. 48). It cannot equal -2.

If an equation involving radicals is not an irrational equation, it can be solved without squaring. For example, $3x^2 - x\sqrt{5} = 0$ should be solved by factoring: $x(3x - \sqrt{5}) = 0$. Hence $x = 0$ or $x = \frac{1}{3}\sqrt{5}$. Is it necessary to check these values to see if any are extraneous?

If an equation involves fractional exponents, it can sometimes be solved by taking the proper power of both sides of the equation.

Example 4. Solve: $(11x - 6)^{\frac{3}{4}} = 8.$

Solution. Take the $\frac{4}{3}$ power of both sides of the equation.

$$[(11x - 6)^{\frac{3}{4}}]^{\frac{4}{3}} = 8^{\frac{4}{3}}.$$
$$11x - 6 = 16.$$
$$11x = 22, \quad x = 2.$$

This value satisfies the given equation.

Exercise 40

Solve.

1. $\sqrt{2x + 3} = 5.$
2. $\sqrt[3]{11x - 2} = 4.$
3. $2x + 4\sqrt{7} = 0.$
4. $2x = 6\sqrt{5}.$
5. $\sqrt{8 - x} = -3.$
6. $\sqrt{x + 7} + 5 = 0.$
7. $x^2 = 3x\sqrt{2}.$
8. $x^2 + 2x\sqrt{11} = 0.$
9. $5\sqrt{x + 1} = 4\sqrt{x + 2}.$
10. $3\sqrt{4x + 3} = 2\sqrt{8x + 7}.$
11. $3\sqrt{x} + 10 = x.$
12. $5\sqrt{x} - 6 = x.$
13. $\sqrt{x + 1} = \sqrt{3x} - 1.$
14. $\sqrt{x + 4} = \sqrt{2x + 1} + 1.$
15. $\sqrt{2x + 5} - \sqrt{2x + 1} = 1.$
16. $\sqrt{3x - 5} = \sqrt{3x - 1} - 1.$
17. $\sqrt{3x + 1} = \sqrt{2x - 1} + 1.$
18. $\sqrt{2x - 2} = \sqrt{x + 6} - 1.$
19. $5\sqrt{x + 1} = 3 + 2\sqrt{4x + 1}.$
20. $\sqrt{4 - 3x} - \sqrt{2 - x} = 2.$
21. $\sqrt{7x + 4} = \sqrt{5x + 1} - 3.$
22. $\sqrt{2x - 5} = \sqrt{5x + 1} + 3.$
23. $\sqrt{6x - 4} = \sqrt{x - 1} + \sqrt{2x}.$
24. $\sqrt{2x + 4} = \sqrt{5x + 7} - \sqrt{x - 1}.$
25. $\sqrt{x + 5} = \sqrt{6x} - \sqrt{x - 1}.$
26. $\sqrt{\sqrt{x + 3} + \sqrt{x - 5}} = 2.$
27. $3\sqrt{x} + \sqrt{x - 4} = 2\sqrt{x + 5}.$
28. $\sqrt{2x + 1} = \sqrt{6x + 2} - \sqrt{2x}.$
29. $\sqrt{2x + 5} + \sqrt{x + 2} = 4.$
30. $x^{\frac{5}{2}} = 32.$
31. $x^{\frac{4}{3}} = 10{,}000.$
32. $(2x + 1)^{\frac{3}{4}} = 27.$
33. $(3x - 1)^{\frac{2}{5}} = 4.$
34. Solve for y: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$

68. Equations in quadratic form. The equation

$$ax^2 + bx + c = 0$$

is said to be a quadratic in x. An equation involving x is said to be a quadratic in y if it can be written in the form

$$ay^2 + by + c = 0,$$

where y is some function of x, and a, b, c do not involve x.

Illustration 1.

$8(x^2 - 3x)^2 + 11(x^2 - 3x) - 10 = 0$ is a quadratic in $(x^2 - 3x)$.
$x^{10} - 17x^5 + 72 = 0$ is a quadratic in x^5 because $x^{10} = (x^5)^2$.
$x^{-6} + x^{-3} - 20 = 0$ is a quadratic in x^{-3} because $x^{-6} = (x^{-3})^2$.

Example 1. Solve for x: $\quad (x^2 - 2x)^2 - 13x^2 + 26x + 40 = 0.$

Solution. Group terms: $(x^2 - 2x)^2 - 13(x^2 - 2x) + 40 = 0$. This is a quadratic in $(x^2 - 2x)$. Let $y = x^2 - 2x$.

$$\begin{aligned} y^2 - 13y + 40 &= 0 \\ (y - 8)(y - 5) &= 0 \\ y = 8 \text{ or } y &= 5. \end{aligned}$$

Since $y = x^2 - 2x$,

$$\left.\begin{aligned} x^2 - 2x &= 8 \\ x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \\ x &= 4, -2. \end{aligned}\;\right|\;\begin{aligned} x^2 - 2x &= 5 \\ x^2 - 2x + 1 &= 6 \\ x - 1 &= \pm\sqrt{6} \\ x &= 1 \pm \sqrt{6}. \end{aligned}$$

The roots are $4, -2, 1 + \sqrt{6}, 1 - \sqrt{6}$.

Example 2. Solve for x: $\quad 2x^{-4} - 17x^{-2} - 9 = 0.$

Solution. This is a quadratic in x^{-2}. Let $y = x^{-2}$. Then $x^{-4} = y^2$.

$$\begin{aligned} 2y^2 - 17y - 9 &= 0. \\ (y - 9)(2y + 1) &= 0. \end{aligned}$$

$$\left.\begin{aligned} y &= 9 \\ x^{-2} &= 9 \\ x^2 &= \tfrac{1}{9} \\ x &= \pm\tfrac{1}{3}. \end{aligned}\;\right|\;\begin{aligned} y &= -\tfrac{1}{2} \\ x^{-2} &= -\tfrac{1}{2} \\ x^2 &= -2 \\ x &= \pm i\sqrt{2}. \end{aligned}$$

The roots are $\frac{1}{3}, -\frac{1}{3}, i\sqrt{2}, -i\sqrt{2}$.

Sometimes a higher degree equation can be solved by factoring.

Example 3. Solve for x: $x^3 = 1000$.

Solution. Transpose to make right side zero: $x^3 - 1000 = 0$.

Factor: $(x - 10)(x^2 + 10x + 100) = 0$.

Set each factor equal to zero:

$$x = 10, \quad x = -5 \pm 5i\sqrt{3}.$$

The third degree equation $x^3 = 1000$ has the three roots:

$$10, \quad -5 + 5i\sqrt{3}, \quad -5 - 5i\sqrt{3}.$$

Exercise 41

Solve for x by reducing to a quadratic equation in some new variable.

1. $x^4 - 15x^2 - 16 = 0$.
2. $x^4 + 2x^2 - 8 = 0$.
3. $x^4 - 81 = 0$.
4. $16x^{-4} - 17x^{-2} + 1 = 0$.
5. $x^{-4} - 10x^{-2} + 9 = 0$.
6. $x^{-2} - 3x^{-1} - 10 = 0$.
7. $x^{\frac{1}{2}} + x^{\frac{1}{4}} - 6 = 0$.
8. $x - \sqrt{x} - 20 = 0$.
9. $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$.
10. $x^{\frac{2}{5}} - 3x^{\frac{1}{5}} + 2 = 0$.
11. $(x^2 - 3x)^2 - 2(x^2 - 3x) = 8$.
12. $(x^4 - 10x^2)^2 - 2(x^4 - 10x^2) - 99 = 0$.
13. $(x^2 - 6x)^2 + 12x^2 - 72x + 35 = 0$.
14. $9(x^2 + 4x)^2 + 2x^2 + 8x = 11$.
15. $\left(\dfrac{x^2 - 12}{2x + 3}\right)^2 + 10\left(\dfrac{x^2 - 12}{2x + 3}\right) = 11$.
16. $\left(x + \dfrac{1}{x}\right)^2 - 9\left(x + \dfrac{1}{x}\right) + 14 = 0$.
17. $\left(\dfrac{x - 1}{x^2}\right)^2 + \left(\dfrac{x - 1}{x^2}\right) = 2$.
18. $\left(2x - \dfrac{5}{x}\right)^2 - 6\left(2x - \dfrac{5}{x}\right) = 27$.
19. $(x^2 + 7x + 4)^2 + 8x^2 + 56x + 44 = 0$.
20. $(x^2 - 3)(x^2 - 4) = 6$.
21. $\dfrac{x^2 - 7}{x + 5} + 2\left(\dfrac{x + 5}{x^2 - 7}\right) = 3$.

Hint. Let $y = \dfrac{x^2 - 7}{x + 5}$; then $\dfrac{x + 5}{x^2 - 7} = \dfrac{1}{y}$.

22. $\sqrt{\dfrac{x+7}{x+1}} + \sqrt{\dfrac{x+1}{x+7}} = \dfrac{5}{2}$.

23. $\dfrac{15}{(x^2-4x)^2} + \dfrac{2}{x^2-4x} = 1$.

24. $\dfrac{4x^2}{(x^2-6)^2} - \dfrac{5x}{x^2-6} + 1 = 0$.

Solve for x by factoring.

25. $x^3 - 10x^2 + 24x = 0$.
26. $5x^3 - 11x^2 + 2x = 0$.
27. $x^3 = 1$.
28. $x^3 + 8 = 0$.
29. $27x^3 + 64 = 0$.
30. $8x^3 = 125$.

69. Character of the roots. When we encounter a quadratic equation, it sometimes happens that we are not interested in the values of the roots but rather in their character. Are the roots real or imaginary, equal or unequal, rational or irrational?

In Art. 64 we proved that *the roots of the equation*

$$ax^2 + bx + c = 0 \tag{1}$$

are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \textit{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The quantity $b^2 - 4ac$ which appears under the radical sign is called the **discriminant** of the quadratic equation (1). If a, b, c, are real numbers, the character of the roots can be determined by merely finding the value of the discriminant. A careful examination of r_1 and r_2 leads to the following conclusions.

If	The roots of $ax^2 + bx + c = 0$ are
$b^2 - 4ac < 0$	**imaginary** and **unequal**
$b^2 - 4ac = 0$	**real** and **equal**
$b^2 - 4ac > 0$	**real** and **unequal**

If a, b, c are rational and $b^2 - 4ac$ is a perfect square or zero, the roots are rational.

Illustrations.

Equation	Discriminant	Character of the roots
$x^2 + 3x + 7 = 0$	$3^2 - 4(1)(7) = -19$	imaginary and unequal
$9x^2 - 30x + 25 = 0$	$(-30)^2 - 4(9)(25) = 0$	real, equal, and rational
$3x^2 - 5x - 2 = 0$	$(-5)^2 - 4(3)(-2) = 49$	real, unequal, and rational
$2x^2 - 7x + 1 = 0$	$(-7)^2 - 4(2)(1) = 41$	real, unequal, and irrational

70. Sum and product of the roots. *The* **sum** *of the roots of the equation* $ax^2 + bx + c = 0$ *is* $-\dfrac{b}{a}$; *the* **product** *of the roots is* $\dfrac{c}{a}$. To prove this, recall that

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Adding, we get

$$r_1 + r_2 = -\frac{b}{2a} - \frac{b}{2a} = -\frac{2b}{2a} = -\frac{b}{a}.$$

Multiplying, we obtain

$$r_1 r_2 = \left(-\frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

These results are useful in checking the roots of a quadratic equation.

Example 1. Without solving, find the sum and product of the roots of $8x^2 = 2x + 3$.

Solution. Write in the standard form $ax^2 + bx + c = 0$:

$$8x^2 - 2x - 3 = 0.$$

$$\text{Sum of roots} = -\frac{b}{a} = -\frac{-2}{8} = \frac{1}{4}.$$

$$\text{Product of roots} = \frac{c}{a} = \frac{-3}{8} = -\frac{3}{8}.$$

Example 2. Given the equation $6x^2 + 2kx - 8x - 3k = 0$, where x is the unknown. Determine the value of k so that:

(*a*) The roots will be equal.
(*b*) The product of the roots will be 10.
(*c*) One root will be the negative of the other.
(*d*) One root will be 3.

Solution. Write the equation in standard form:

$$6x^2 + (2k - 8)x - 3k = 0.$$

Then $a = 6, \quad b = 2k - 8, \quad c = -3k.$

(*a*) The roots will be equal if $b^2 - 4ac = 0$.

Discriminant $= (2k - 8)^2 - 4(6)(-3k) = 0.$

$$\begin{aligned} 4k^2 + 40k + 64 &= 0 \\ k^2 + 10k + 16 &= 0 \\ k = -2 \quad \text{or} \quad k &= -8. \end{aligned}$$

(*b*) Product $= \dfrac{c}{a} = \dfrac{-3k}{6} = 10, \quad k = -20.$

(*c*) If one root is the negative of the other, their sum must be zero. Hence $-\dfrac{b}{a} = -\dfrac{2k - 8}{6} = 0, \quad k = 4.$

(*d*) If 3 is to be a root, the equation must be satisfied by $x = 3$:

$$6(3)^2 + (2k - 8)3 - 3k = 0, \quad k = -10.$$

The student should check these results by substituting the various values of k in the original equation and then finding the roots of the several equations.

71. Forming an equation when the roots are given. *If the roots of a quadratic equation are r_1 and r_2, the equation can be written in the form*

$$(x - r_1)(x - r_2) = 0, \tag{1}$$

or

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0. \tag{2}$$

The truth of this statement can be established by solving the first equation.

Example 1. Find a quadratic equation whose roots are $\frac{1}{2}$ and $-\frac{4}{3}$.

Solution. $$(x - \tfrac{1}{2})(x + \tfrac{4}{3}) = 0.$$

Multiply through by $2 \cdot 3$ to clear of fractions.

$$2(x - \tfrac{1}{2})3(x + \tfrac{4}{3}) = 0.$$
$$(2x - 1)(3x + 4) = 0.$$
$$6x^2 + 5x - 4 = 0.$$

Example 2. Find a quadratic equation whose roots are $3 + \sqrt{5}$ and $3 - \sqrt{5}$.

First solution. $$[x - (3 + \sqrt{5})][x - (3 - \sqrt{5})] = 0.$$
$$[\overline{x - 3} - \sqrt{5}][\overline{x - 3} + \sqrt{5}] = 0.$$
$$(x - 3)^2 - (\sqrt{5})^2 = 0.$$
$$x^2 - 6x + 4 = 0.$$

Second solution. $$r_1 + r_2 = (3 + \sqrt{5}) + (3 - \sqrt{5}) = 6.$$
$$r_1 r_2 = (3 + \sqrt{5})(3 - \sqrt{5}) = 9 - 5 = 4.$$

Substitution in (2) gives us $x^2 - 6x + 4 = 0$.

Exercise 42

Use the discriminant to determine the character of the roots. Do not solve.

1. $9x^2 - 3x - 2 = 0$.
2. $2x^2 - x + 5 = 0$.
3. $4x^2 - 20x + 25 = 0$.
4. $3x^2 - 6x + 1 = 0$.
5. $5x^2 = 2x - 3$.
6. $4x^2 + 6 = 11x$.
7. $-x^2 + 3x + 7 = 0$.
8. $25x^2 = 40x - 16$.
9. $5x^2 = 7$.
10. $7x^2 = 2x$.

Find the sum and the product of the roots without solving.

11. $3x^2 + 10x + 8 = 0$.
12. $4x^2 - x - 14 = 0$.
13. $-2x^2 - 7x + 4 = 0$.
14. $-6x^2 + 7x - 8 = 0$.
15. $5x^2 - 11 = 0$.
16. $8x^2 = 9x$.
17. $x^2 = 3x + 4$.
18. $6x = 7 - 2x^2$.

Form a quadratic equation that has the following roots.

19. 7, 9.
20. -6, 0.
21. 8, -3.
22. 5, 5.
23. 4, $-\frac{2}{5}$.
24. $\frac{3}{4}$, $-\frac{5}{6}$.

25. $-\frac{1}{2}, -\frac{3}{7}$.

26. $\pm 3\sqrt{2}$.

27. $5 \pm \sqrt{6}$.

28. $\pm 4i$.

29. $-7 \pm i$.

30. $\dfrac{2 \pm 5i}{3}$.

31. $\dfrac{3 \pm i\sqrt{5}}{2}$.

32. $\dfrac{-4 \pm 3i\sqrt{2}}{5}$.

33. Given the equation $2x^2 - kx^2 + 4x - 1 = 0$, where x is the unknown. Determine the value of k so that:
(*a*) The roots will be equal.
(*b*) The sum of the roots will be $-\frac{4}{5}$.
(*c*) The product of the roots will be $\frac{4}{15}$.
(*d*) One root will be 1.

34. Given the equation $4x^2 - 2kx + k = 0$, where x is the unknown. Determine the value of k so that:
(*a*) The roots will be equal.
(*b*) The sum of the roots will be $\frac{9}{4}$.
(*c*) The product of the roots will be 7.
(*d*) One root will be 2.

35. Given the equation $2x^2 - 4x - k = 0$, where x is the unknown. Determine the value of k so that:
(*a*) The roots will be equal.
(*b*) One root will be 10.
(*c*) One root exceeds the other by 10.
(*d*) One root is five times the other.

36. Given the equation $3x^2 + 6kx + k + 2 = 0$, where x is the unknown. Determine the value of k so that:
(*a*) The roots will be equal.
(*b*) One root will be 0.
(*c*) One root is the reciprocal of the other.
(*d*) The sum of the roots will be equal to the product of the roots.

72. Graph of a quadratic function. A quadratic function of x is an expression of the form $ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$.

The student should review Art. 35 before proceeding.

Example 1. Graph the following quadratic function of x:

$$-x^2 + 4x + 2.$$

Solution. Let y represent the function of x:

$$y = -x^2 + 4x + 2.$$

Paragraph 2 of Art. 35 suggests a procedure in selecting the values to be assigned to x.

x	-1	0	1	2	3	4	5
y	-3	2	5	6	5	2	-3

The graph of $-x^2 + 4x + 2$ is shown in Fig. 13.

Example 2. Graph the following quadratic function of x:

$$3x^2 - 6x - 4.$$

Solution. Let y represent the function of x:

$$y = 3x^2 - 6x - 4.$$

The student should prepare a table of values and verify the graph shown in Fig. 14.

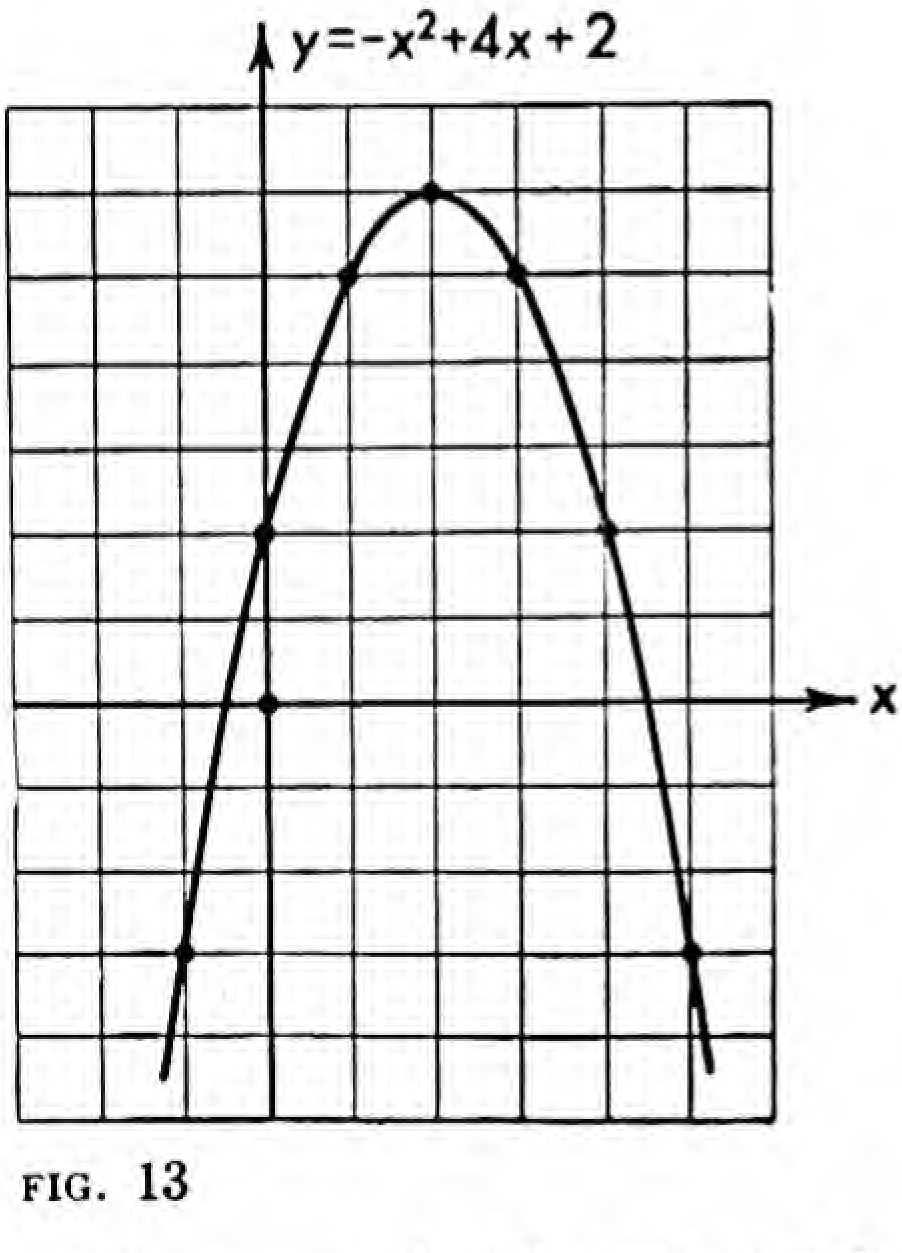


FIG. 13

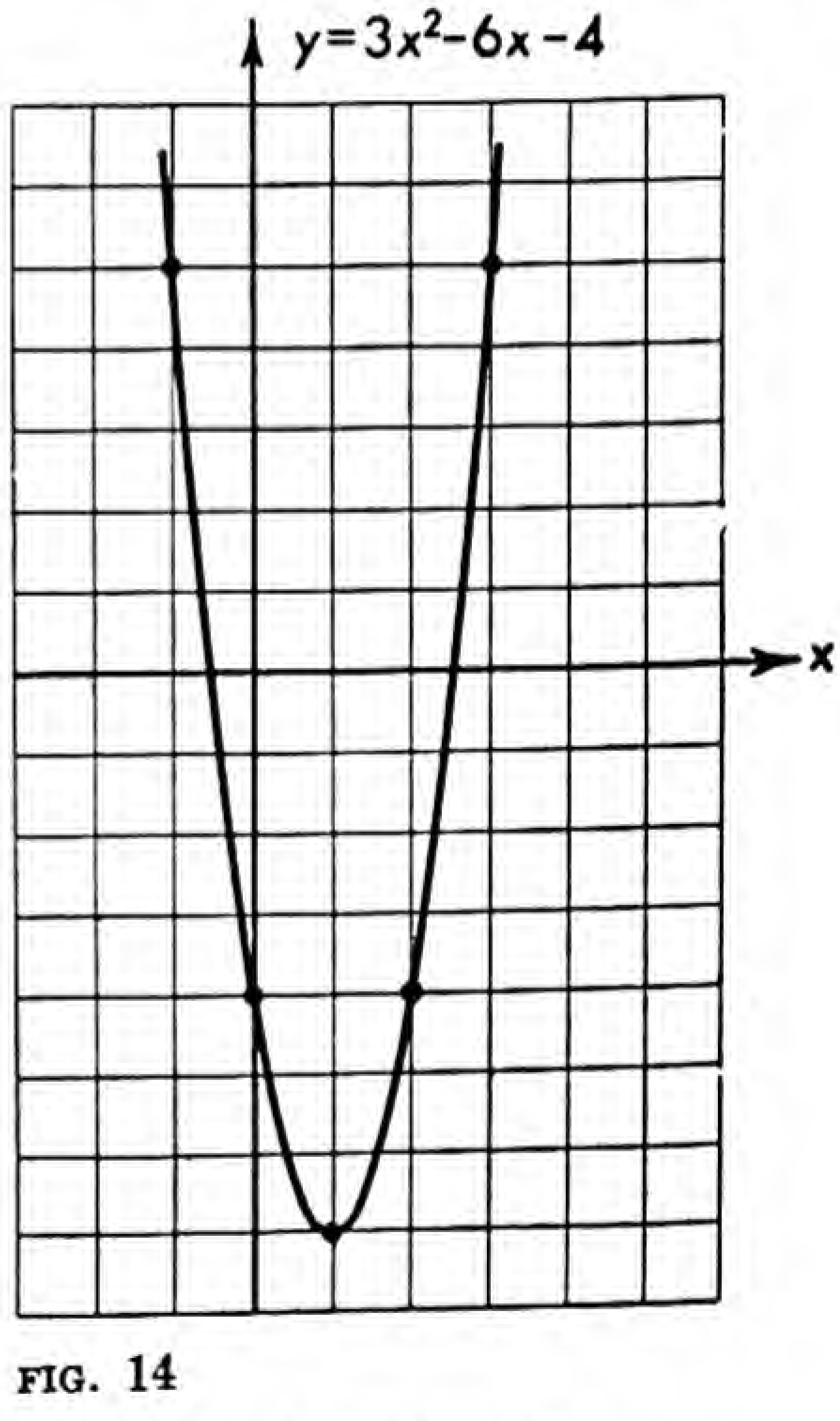


FIG. 14

The curves in Fig. 13 and Fig. 14 are called **parabolas.**

In analytic geometry it is proved that

The graph of the quadratic function $ax^2 + bx + c$ is a parabola that opens upward if a is positive, and downward if a is negative.

73. Graphic solution of a quadratic equation.

Example 1. Solve graphically: $3x^2 = 6x + 4$.

Solution. Make the right side equal to zero by transposing terms:

$$3x^2 - 6x - 4 = 0. \tag{1}$$

Let

$$y = 3x^2 - 6x - 4 \tag{2}$$

and then graph this function (Fig. 14). From the curve we see that y is 0 (the curve crosses the x-axis) when x is $-.5$ and 2.5, approximately. Notice that when $y = 0$, equation (2) reduces to equation (1). Hence the graphic solution of the given equation yields the roots $-.5$ and 2.5, approximately.

Since the graphic solution of a quadratic usually gives only approximations for the real roots, it is generally better to solve algebraically. The graphic method is discussed only because it indicates a means of approximating the roots of higher degree equations.

Illustration 1 and Fig. 15 exemplify the three possibilities that may occur in solving a quadratic graphically.

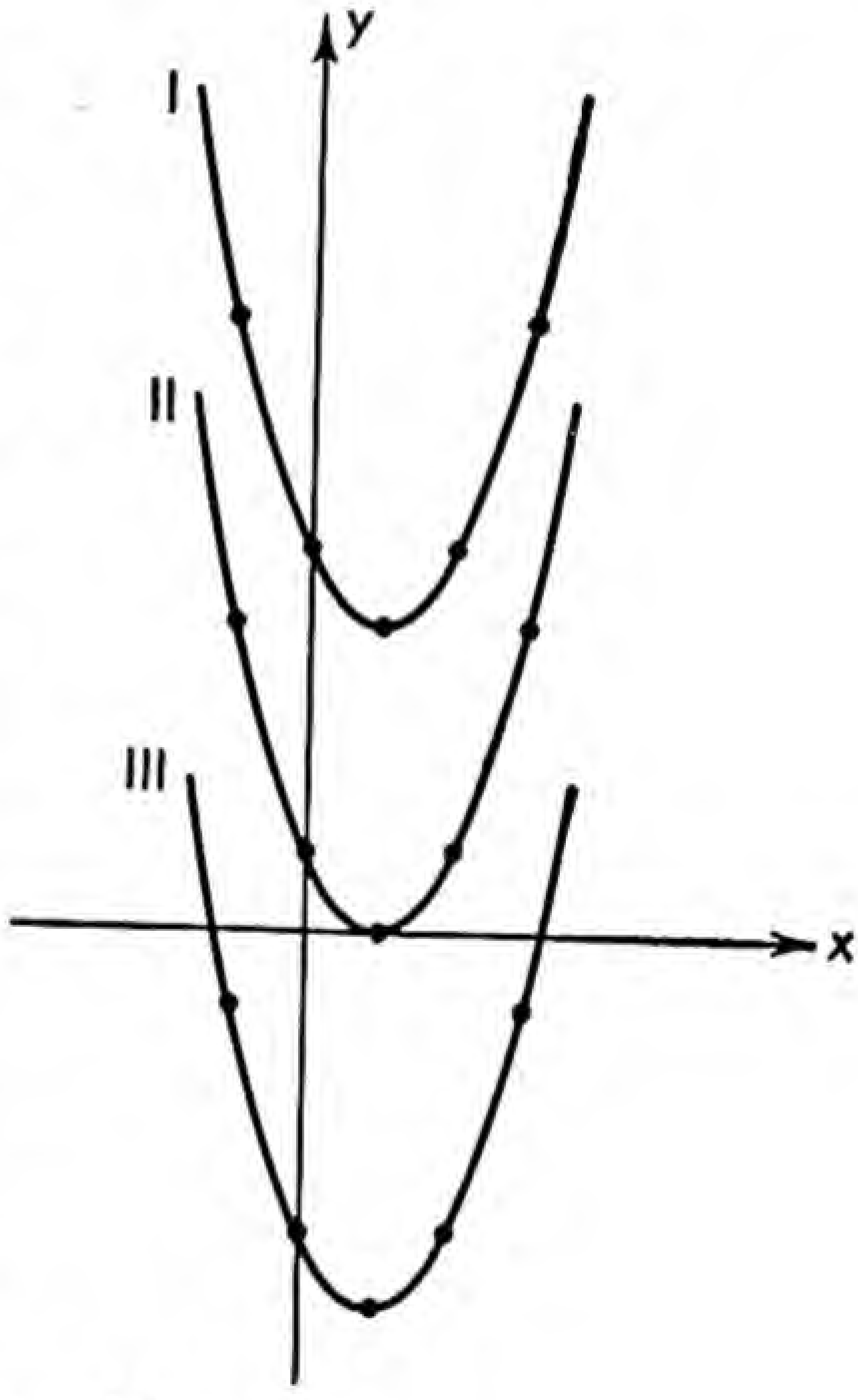


FIG. 15

Illustration 1.

	Equation	Discriminant	Roots	Graph of left side of equation
I	$x^2 - 2x + 5 = 0$	-16	imaginary	does not meet x-axis
II	$x^2 - 2x + 1 = 0$	0	real, equal	is tangent to x-axis
III	$x^2 - 2x - 4 = 0$	20	real, unequal	cuts x-axis in two points

74. Maximum or minimum value of a quadratic function. In Fig. 13, the highest point on the parabola is (2, 6). We say that the function $-x^2 + 4x + 2$ has a **maximum** value of 6 when $x = 2$. The function $3x^2 - 6x - 4$, whose graph is shown in Fig. 14, has a **minimum** value of -7 when $x = 1$. The maximum or minimum value of a quadratic function can be found by completing the square.

Example 1. Find the minimum value of $5x^2 - 6x + 4$.

Solution.

Let

$$\begin{aligned} y &= 5x^2 - 6x + 4 \\ &= 5(x^2 - \tfrac{6}{5}x) + 4 \\ &= 5(x^2 - \tfrac{6}{5}x + \tfrac{9}{25}) + 4 - \tfrac{9}{5} \\ &= 5(x - \tfrac{3}{5})^2 + \tfrac{11}{5}. \end{aligned}$$

For real values of x, the expression $5(x - \frac{3}{5})^2$ is always positive or zero. Its minimum value of zero occurs when $x = \frac{3}{5}$. Hence the given function $5x^2 - 6x + 4$ has a minimum value of $\frac{11}{5}$ when $x = \frac{3}{5}$.

Example 2. A theater operator discovered that when he increased the admission charge from 40 cents to 45 cents, the average daily attendance dropped from 1200 to 1100. Find the admission charge that will produce the maximum daily income, assuming that each increase of 1 cent in price will result in a decrease of 20 in daily attendance.

Solution.

Let x = number of cents the original price should be increased.

Then $40 + x$ = number of cents in the new admission charge.

And $1200 - 20x$ = average daily attendance.

If I designates the average daily income (in cents),

$$\begin{aligned} I &= (40 + x)(1200 - 20x) \\ &= -20x^2 + 400x + 48{,}000 \\ &= -20(x^2 - 20x) + 48{,}000 \\ &= -20(x^2 - 20x + 100) + 48{,}000 + 2{,}000 \\ &= -20(x - 10)^2 + 50{,}000. \end{aligned}$$

The expression $-20(x - 10)^2$ is always negative or zero. Its maximum value of zero occurs when $x = 10$. Hence I has a maximum value of 50,000 cents or \$500 when $x = 10$. The admission charge should be increased to 50 cents.

Exercise 43

Graph the following quadratic functions of x.

1. $x^2 - 2x - 2$. **2.** $\frac{1}{2}x^2$. **3.** $\frac{1}{3}x^2 - 4$.

4. $x^2 - 2x + 1$. **5.** $2x^2 + 8x$. **6.** $x^2 + x + 2$.

7. $x^2 + 2x + 1$. **8.** $-2x^2$. **9.** $9 - x^2$.

10. $- x^2 + 4x - 4$. **11.** $- x^2 - 4x + 1$. **12.** $- 2x^2 - 3x$.

Solve graphically.

13. $x^2 - x - 6 = 0$. **14.** $x^2 + 2x - 8 = 0$.

15. $x^2 + 3x - 7 = 0$. **16.** $x^2 = 4x - 2$.

17. $2x^2 + x = 4$. **18.** $- x^2 + 6x - 8 = 0$.

Use the discriminant to determine if the graph of the function cuts the x-axis in two points, or is tangent to the x-axis, or does not meet the x-axis. Do not graph the function.

19. $x^2 + 12x + 36$. **20.** $3x^2 + 7x + 2$. **21.** $5x^2 + x + 2$.

22. $9x^2 - 30x + 25$. **23.** $2x^2 - 3x - 4$. **24.** $x^2 - 2x + 6$.

Find the maximum or minimum value of each of the following quadratic functions of x. State the corresponding value of x and tell whether it gives a maximum or minimum.

25. $3x^2 + 6x - 1$. **26.** $- 2x^2 - 10x + 3$.

27. $- x^2 + 8x + 3$. **28.** $x^2 - 6x + 2$.

29. Find two numbers whose sum is 20 and whose product is a maximum.

30. Find two numbers whose sum is 20 so that the sum of the squares of the numbers is a minimum.

31. Find the dimensions of the rectangle of maximum area that can be enclosed by 100 feet of fence.

32. One side of a rectangular plot of land is adjacent to a river; 400 feet of fencing are available for the other three sides. Find the dimensions of the plot having the largest area.

33. A plot of land forms a right triangle whose legs are 80 feet and 100 feet. Find the dimensions of the rectangular building with maximum floor area that can be built on the lot if the structure is to face the 100-foot side.

34. A man owns a large building containing 80 apartments. If he charges $40 per month rent for each apartment, he can keep them all rented. For each dollar added to the rent, he will have one vacancy. What should he charge to obtain the maximum return?

chapter 9 *Systems of equations involving quadratics*

75. Quadratic equations in two unknowns. The **degree of a term** involving several letters is the sum of the exponents appearing on those letters. (The exponents must be positive integers.) For example, $5x^2y$ is of degree 3; $8xy^2z^4$ is of degree 7; $-9xy$ is of second degree. (See Art. 17.)

The **degree of an equation** is the same as that of its term (or terms) of highest degree. Thus, $x^2 + xy^3 + 7y = 8$ is an equation of degree 4.

A **quadratic equation** is an equation of second degree.

The general quadratic equation in x and y is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

where A, B, C, D, E, and F are constants and at least one of the coefficients A, B, C is not zero. The second-degree terms are Ax^2, Bxy, and Cy^2. The first-degree terms are Dx and Ey. The constant term F is of degree zero.

76. One linear and one quadratic equation. A system consisting of one linear equation and one quadratic equation can always be solved by elimination by substitution. Solve the linear equation for one unknown in terms of the other and then substitute in the quadratic equation.

Example 1. Solve the following linear-quadratic system:

$$\begin{cases} 2x + 3y = 1, & (1) \\ x^2 - 5xy - 8y^2 + 6y = 0. & (2) \end{cases}$$

Solution. Solve the linear equation for x in terms of y:

$$x = \frac{1 - 3y}{2}. \tag{3}$$

Substitute in the quadratic equation:

$$\left(\frac{1-3y}{2}\right)^2 - 5\left(\frac{1-3y}{2}\right)y - 8y^2 + 6y = 0$$

$$\frac{1 - 6y + 9y^2}{4} - \frac{5y - 15y^2}{2} - 8y^2 + 6y = 0$$

$$1 - 6y + 9y^2 - 10y + 30y^2 - 32y^2 + 24y = 0$$

$$7y^2 + 8y + 1 = 0$$

$$(y + 1)(7y + 1) = 0; \quad y = -1, -\tfrac{1}{7}.$$

Substitute in the linear equation (3). For $y = -1$, $x = \dfrac{1+3}{2} = 2.$

For $y = -\frac{1}{7}$, $x = \dfrac{1 + \frac{3}{7}}{2} = \frac{5}{7}.$

The solutions are

$$(x = 2, y = -1) \quad \text{and} \quad (x = \tfrac{5}{7}, y = -\tfrac{1}{7}).$$

Each solution should be checked by substitution in both of the given equations.

In choosing the unknown to be eliminated, it is advisable to avoid fractions if possible. For example, the linear equation $6x - y = 7$ should be solved for y in terms of x: $y = 6x - 7$, rather than for x in terms of y: $x = \dfrac{y+7}{6}.$

Exercise 44

Solve.

1. $\begin{cases} x + y = 3, \\ 2x^2 - y^2 - 10x + 12 = 0. \end{cases}$

2. $\begin{cases} 2x + y = 3, \\ 3x^2 + xy + y^2 = 9. \end{cases}$

3. $\begin{cases} x + 2y = 5, \\ x^2 - xy - y^2 = 5. \end{cases}$

4. $\begin{cases} x - 5y - 2 = 0, \\ x^2 - 3xy - 11y^2 - 8y + 3 = 0. \end{cases}$

5. $\begin{cases} 2x - y + 3 = 0, \\ 4x^2 + xy - y^2 + 19x - 3 = 0. \end{cases}$

6. $\begin{cases} x - 2y = 7, \\ xy = 4. \end{cases}$

7. $\begin{cases} 3x + y = 1, \\ 5x^2 + 4xy + y^2 + x = 2. \end{cases}$

8. $\begin{cases} 2x - y - 2 = 0, \\ x^2 - y - 5 = 0. \end{cases}$

9. $\begin{cases} x - 3y + 2 = 0, \\ x^2 - 2xy - y^2 + 6 = 0. \end{cases}$

10. $\begin{cases} x + y = 4, \\ x^2 - 3xy + 5y^2 - 2x = 62. \end{cases}$

11. $\begin{cases} 3x + 4y = 12, \\ xy + 6y = 12. \end{cases}$

12. $\begin{cases} 2x - 3y = 1, \\ x^2 - xy - y^2 + 1 = 0. \end{cases}$

13. $\begin{cases} 3x - 2y = 1, \\ x^2 + 6xy - 4y^2 - 3y = 0. \end{cases}$

14. $\begin{cases} 4x + 5y = 0, \\ x^2 - xy - 3y^2 + 2x + y = 0. \end{cases}$

15. $\begin{cases} 2x - 5y = 1, \\ 4x^2 - 9xy - 2y^2 - 8x + 15y + 3 = 0. \end{cases}$

16. $\begin{cases} 2x + 3y = 1, \\ 2x^2 - 5xy - 9y^2 + 7y - 2 = 0. \end{cases}$

17. $\begin{cases} 5x + 3y = 8, \\ x^2 - xy - y^2 - x + 3y + 1 = 0. \end{cases}$

18. $\begin{cases} 3x + 5y = 7, \\ x^2 + 2xy + y^2 - 6x + 5 = 0. \end{cases}$

19. $\begin{cases} xy = 4, \\ x^2 - y^2 = 15. \end{cases}$

Hint. Solve the first equation for y in terms of x and substitute in the second.

20. $\begin{cases} xy = 6, \\ x^2 - xy + 2y^2 = 12. \end{cases}$

77. Two quadratic equations. If we attempt to solve a system of two quadratic equations by eliminating one unknown, the result is usually a fourth-degree equation in the other unknown.

Illustration. Solve $\begin{cases} x^2 + 2xy - 2y^2 + x - y + 3 = 0, \\ x^2 + xy - y^2 - 4y + 5 = 0. \end{cases}$

Subtract to eliminate x^2: $xy - y^2 + x + 3y - 2 = 0$.

Solve for x: $$x = \frac{y^2 - 3y + 2}{y + 1}.$$

Substituting in the first equation and simplifying, we get

$$y^4 - 14y^3 + 8y^2 - 4y + 9 = 0.$$

At present we have not learned how to solve equations of degree higher than two. Accordingly, in this chapter we shall confine ourselves to certain special cases (Arts. 78, 79, 80) which are solvable by quadratics.

A system of two quadratic equations ordinarily has four solutions, some or all of which may be imaginary.

78. If one unknown can be eliminated by addition or subtraction, the system can be reduced to one equation, either linear or quadratic, involving only one unknown.

Example 1. Solve: $\begin{cases} 4xy + 2y^2 - 3y - 12 = 0, & (1) \\ 6xy + 2y^2 - 5y - 13 = 0. & (2) \end{cases}$

Solution. We shall eliminate x.

$$\begin{array}{ll} \text{Multiply (1) by 3:} & 12xy + 6y^2 - 9y - 36 = 0. \\ \text{Multiply (2) by 2:} & 12xy + 4y^2 - 10y - 26 = 0. \\ \text{Subtract:} & 2y^2 + y - 10 = 0. \end{array}$$

$$(y - 2)(2y + 5) = 0; \quad y = 2, -\tfrac{5}{2}.$$

Substitute in either (1) or (2). If $y = 2$, $x = \frac{5}{4}$. If $y = -\frac{5}{2}$, $x = \frac{4}{5}$.

The solutions are

$$(x = \tfrac{5}{4}, y = 2) \quad \text{and} \quad (x = \tfrac{4}{5}, y = -\tfrac{5}{2}),$$

which may be indicated by $(\frac{5}{4}, 2)$; $(\frac{4}{5}, -\frac{5}{2})$.

The student should demonstrate that each solution satisfies both of the given equations.

Exercise 45

Solve.

1. $\begin{cases} 4x^2 + y^2 = 41, \\ 4x^2 - y^2 = 9. \end{cases}$

2. $\begin{cases} x^2 + 3y^2 = 17, \\ x^2 + 2y^2 = 15. \end{cases}$

3. $\begin{cases} 5x^2 - 6y^2 + 21 = 0, \\ 4x^2 - 7y^2 + 8 = 0. \end{cases}$

4. $\begin{cases} 2x^2 + 7y^2 - 13 = 0, \\ 3x^2 - 4y^2 - 34 = 0. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 36, \\ y = x^2 - 6. \end{cases}$

6. $\begin{cases} x^2 + y^2 - 13 = 0, \\ x^2 - y - 7 = 0. \end{cases}$

7. $\begin{cases} x^2 + 4y^2 = 24, \\ 2y^2 + x = 12. \end{cases}$

8. $\begin{cases} 2x^2 - y = 1, \\ 5x^2 + 2y = 2. \end{cases}$

9. $\begin{cases} 2x^2 + 5xy = 8, \\ 4x^2 + 7xy - x = 11. \end{cases}$

10. $\begin{cases} 3xy + 2y^2 + 4y = 9, \\ 5xy + y^2 + 2y = 8. \end{cases}$

11. $\begin{cases} 7xy + 8y^2 + 33 = 0, \\ 2xy + 3y^2 + 3 = 0. \end{cases}$

12. $\begin{cases} x^2 + 6xy = 16, \\ 5x^2 - 8xy = 4. \end{cases}$

13. $\begin{cases} 3x^2 - y^2 - 3y + 7 = 0, \\ 5x^2 - 2y^2 - 4y + 13 = 0. \end{cases}$

14. $\begin{cases} x^2 - y^2 - 3x + 1 = 0, \\ 4x^2 - 5y^2 - 6x - 15 = 0. \end{cases}$

15. $\begin{cases} x^2 - 6xy + 2x + 3y - 8 = 0, \\ x^2 - 8xy + x + 4y + 6 = 0. \end{cases}$

16. $\begin{cases} 3x^2 + 6xy - 7y^2 + 4y = 6, \\ 2x^2 + 4xy - 5y^2 + 3y = 4. \end{cases}$

79. If all terms involving the unknowns are of second degree, we eliminate the constant terms and then solve two linear-quadratic systems.

Example 1. Solve: $\begin{cases} 5x^2 + 7xy - 4y^2 = 10, & (1) \\ 11x^2 + 15xy - 9y^2 = 25. & (2) \end{cases}$

Solution. We shall eliminate constants.

Multiply (1) by 5:	$25x^2 + 35xy - 20y^2 = 50.$	
Multiply (2) by 2:	$22x^2 + 30xy - 18y^2 = 50.$	
Subtract:	$3x^2 + 5xy - 2y^2 = 0.$	(3)
Factor: *	$(x + 2y)(3x - y) = 0.$	

$x + 2y = 0.$	$3x - y = 0.$	
$x = -2y.$	$y = 3x.$	(4)
Substitute in (1): †		
$20y^2 - 14y^2 - 4y^2 = 10.$	$5x^2 + 21x^2 - 36x^2 = 10.$	
$y^2 = 5.$	$x^2 = -1.$	
$y = \pm\sqrt{5}.$	$x = \pm i.$	
$y = \sqrt{5}.$ $y = -\sqrt{5}.$	$x = i.$ $x = -i.$	
Substitute in (4):		
$x = -2\sqrt{5}.$ $x = 2\sqrt{5}.$	$y = 3i.$ $y = -3i.$	

The solutions are $(-2\sqrt{5}, \sqrt{5})$, $(2\sqrt{5}, -\sqrt{5})$, $(i, 3i)$, $(-i, -3i)$. Each of them satisfies both of the original equations.

If one equation of the system contains no constant term, we solve this equation for x in terms of y, as in (3) of Ex. 1, and then substitute in the other equation.

80. Symmetric equations. An equation is **symmetric in x and y** if the equation is unaltered when x and y are interchanged. The following quadratic equation is symmetric in x and y:

$$5x^2 + 6xy + 5y^2 - 7x - 7y + 8 = 0.$$

A system of two quadratic equations that are symmetric in x and y can be solved by setting x equal to $u + v$, and y equal to $u - v$.

* If the expression is not factorable, solve for x in terms of y by use of the quadratic formula or by completing the square.

† Or in (2).

Example 1. Solve: $\begin{cases} 2xy - 13x - 13y + 56 = 0, & (1) \\ 3x^2 + 2xy + 3y^2 + 8x + 8y - 88 = 0. & (2) \end{cases}$

Solution. Let $x = u + v, \quad y = u - v.$

Then (1) becomes $u^2 - v^2 - 13u + 28 = 0.$ (3)

And (2) becomes $2u^2 + v^2 + 4u - 22 = 0.$ (4)

Eliminate v from (3) and (4) by addition:

$$3u^2 - 9u + 6 = 0.$$

Divide by 3:

$$u^2 - 3u + 2 = 0.$$

$$(u - 1)(u - 2) = 0; \quad u = 1, 2.$$

Substituting $u = 1$ in (3) or (4), we get $v = \pm 4$. Similarly $u = 2$ gives $v = \pm\sqrt{6}$. Hence the solutions of the system [(3), (4)] are

$$\begin{cases} u = 1, \\ v = 4. \end{cases} \quad \begin{cases} u = 1, \\ v = -4. \end{cases} \quad \begin{cases} u = 2, \\ v = \sqrt{6}. \end{cases} \quad \begin{cases} u = 2, \\ v = -\sqrt{6}. \end{cases}$$

Recalling that $x = u + v$ and $y = u - v$, we see that the solutions of the given system are

$$(x = 5, y = -3), \quad (x = -3, y = 5),$$
$$(x = 2 + \sqrt{6}, y = 2 - \sqrt{6}), \quad (x = 2 - \sqrt{6}, y = 2 + \sqrt{6}).$$

Notice that the solutions are symmetric in x and y.

Exercise 46

Solve.

1. $\begin{cases} x^2 + 2xy + 2y^2 = 20, \\ 2x^2 + 6xy + 7y^2 = 60. \end{cases}$

2. $\begin{cases} x^2 - 3xy - y^2 = -12, \\ 2x^2 - 7xy = -36. \end{cases}$

3. $\begin{cases} x^2 - 2xy + 3y^2 = 12, \\ x^2 + xy = 24. \end{cases}$

4. $\begin{cases} x^2 + xy = 4, \\ xy - y^2 = -6. \end{cases}$

5. $\begin{cases} 2x^2 - 4xy + y^2 = 2, \\ x^2 - 7xy + 2y^2 = 5. \end{cases}$

6. $\begin{cases} 6x^2 + xy + y^2 = 4, \\ 17x^2 + xy + 2y^2 = 8. \end{cases}$

7. $\begin{cases} 9x^2 + xy - y^2 = 12, \\ 3x^2 - 2xy + 2y^2 = 60. \end{cases}$

8. $\begin{cases} x^2 + xy + 2y^2 = 4, \\ x^2 + 3y^2 = 7. \end{cases}$

9. $\begin{cases} 4x^2 + xy - y^2 = 8, \\ 4x^2 - 7xy + 3y^2 = 0. \end{cases}$

10. $\begin{cases} 2x^2 + xy - 4y^2 = 18, \\ 4x^2 + 3xy - 10y^2 = 0. \end{cases}$

11. $\begin{cases} x^2 + y^2 - 3x - 3y = -2, \\ 2xy - x - y = 2. \end{cases}$

12. $\begin{cases} 3x^2 + 2xy + 3y^2 = 28, \\ 2xy - 3x - 3y = -14. \end{cases}$

13. $\begin{cases} x^2 + y^2 - 8x - 8y = -22, \\ xy - 4x - 4y = -13. \end{cases}$

14. $\begin{cases} x^2 + y^2 - 3x - 3y + 2 = 0, \\ 2xy - 5x - 5y + 10 = 0. \end{cases}$

15. $\begin{cases} 3x^2 - 4xy + 3y^2 = 22, \\ x^2 + y^2 - 3xy - 3x - 3y = 3. \end{cases}$

16. $\begin{cases} 5x^2 - 2xy + 5y^2 + x + y = 24, \\ 6x^2 + 6y^2 - 11x - 11y = -3. \end{cases}$

81. Graphic solution. In order to graph a quadratic equation in x and y, solve the equation for one variable in terms of the other and then proceed as in Art. 35.

The graphic solution of a system involving quadratic equations is obtained by graphing both equations on the same coordinate system and then estimating the coordinates of the points of intersection of the graphs. Only real solutions appear on the graph and these will usually be only approximations. A point of tangency indicates a double solution. Solutions involving imaginary numbers cannot be obtained graphically.

Example 1. Solve graphically: $\begin{cases} x + 3y = 3, & (1) \\ 2x^2 + y^2 = 18. & (2) \end{cases}$

Solution. Equation (1) is of the first degree. Its graph is a straight line (Art. 38) passing through (0, 1) and (3, 0).

In equation (2), set $x = 0$ and find $y = \pm\sqrt{18} = \pm 4.2$. Then set $y = 0$ and find $x = \pm 3$. The four solutions $(0, \sqrt{18})$, $(0, -\sqrt{18})$, $(3, 0)$, $(-3, 0)$ represent four important points on the curve. To obtain other points, solve for y: $y = \pm\sqrt{18 - 2x^2}$. Assign values to x and compute corresponding values for y. Such pairs of values are given in the following table.

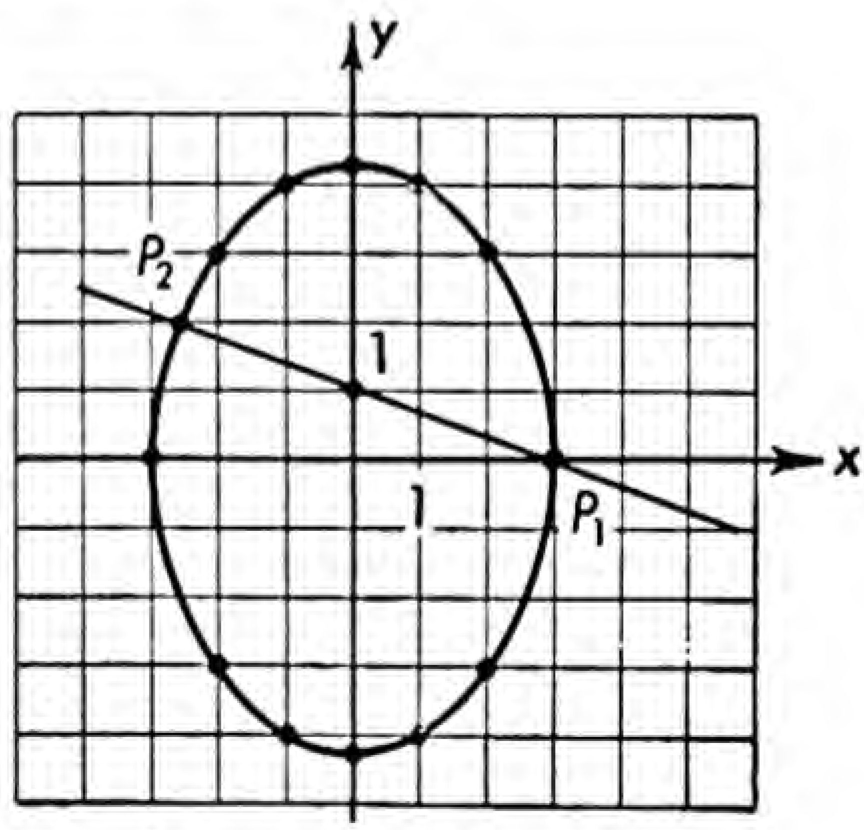


FIG. 16

x		-4	-3	-2	-1	0	1	2	3	4	
$y = \pm\sqrt{18 - 2x^2}$		imag.	0	± 3.2	± 4	± 4.2	± 4	± 3.2	0	imag.	

The resulting oval-shaped curve is called an **ellipse.** The straight line and the ellipse are shown in Fig. 16.

Approximating the coordinates of the points of intersection, we find $P_1(x = 3, y = 0)$ and $P_2(x = -2.7, y = 1.9)$.

Example 2. Solve graphically: $\begin{cases} x^2 + y^2 = 13. & (1) \\ \quad\; y = x^2 - 1. & (2) \end{cases}$

Solution. In equation (1), set $x = 0$ and find $y = \pm\sqrt{13} = \pm 3.6$. Then set $y = 0$ and find $x = \pm\sqrt{13}$. To get additional points, solve for y and prepare a table of values.

x		-4	-3.6	-3	-2	-1	0	1	2	3	3.6	4	
$y = \pm\sqrt{13 - x^2}$		imag.	0	± 2	± 3	± 3.5	± 3.6	± 3.5	± 3	± 2	0	imag.	

The curve is a **circle.**

In equation (2), y is a quadratic function of x; hence the graph is a parabola (Art. 72).

x		-3	-2	-1	0	1	2	3	
$y = x^2 - 1$		8	3	0	-1	0	3	8	

From the graphs (Fig. 17) we read the real solutions $(x = 2, y = 3)$ and $(x = -2, y = 3)$.

The algebraic solution (Art. 78) gives $(x = 2,\ y = 3)$, $(x = -2,\ y = 3)$, $(x = i\sqrt{3},\ y = -4)$, $(x = -i\sqrt{3},\ y = -4)$. Notice that the imaginary solutions do not appear on the graph.

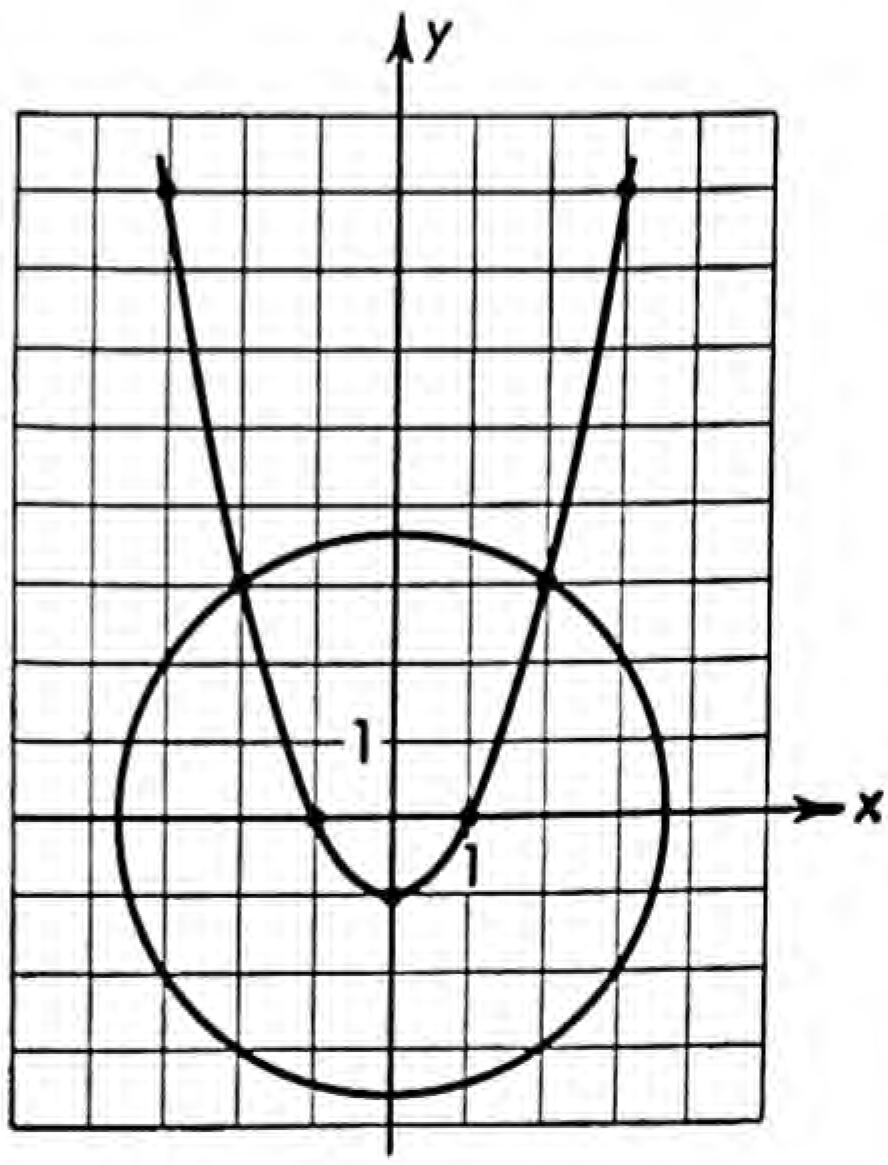


FIG. 17

Example 3. Graph the equation $x^2 - 2y^2 = 4$.

Solution. Set $x = 0$ and find $-2y^2 = 4$, $y^2 = -2$, $y = \pm i\sqrt{2}$. This means that the curve does not meet the y-axis. Set $y = 0$ and find $x = \pm 2$. To avoid fractions and negative radicands, we solve for x in terms of y rather than

for y in terms of x: $x = \pm\sqrt{2y^2 + 4}$. Assigning values to y and computing corresponding values of x, we get the following table.

$x = \pm\sqrt{2y^2+4}$		± 4.7	± 3.5	± 2.4	± 2	± 2.4	± 3.5	± 4.7	
y		-3	-2	-1	0	1	2	3	

The curve is called a **hyperbola** and consists of two parts or **branches** (Fig. 18).

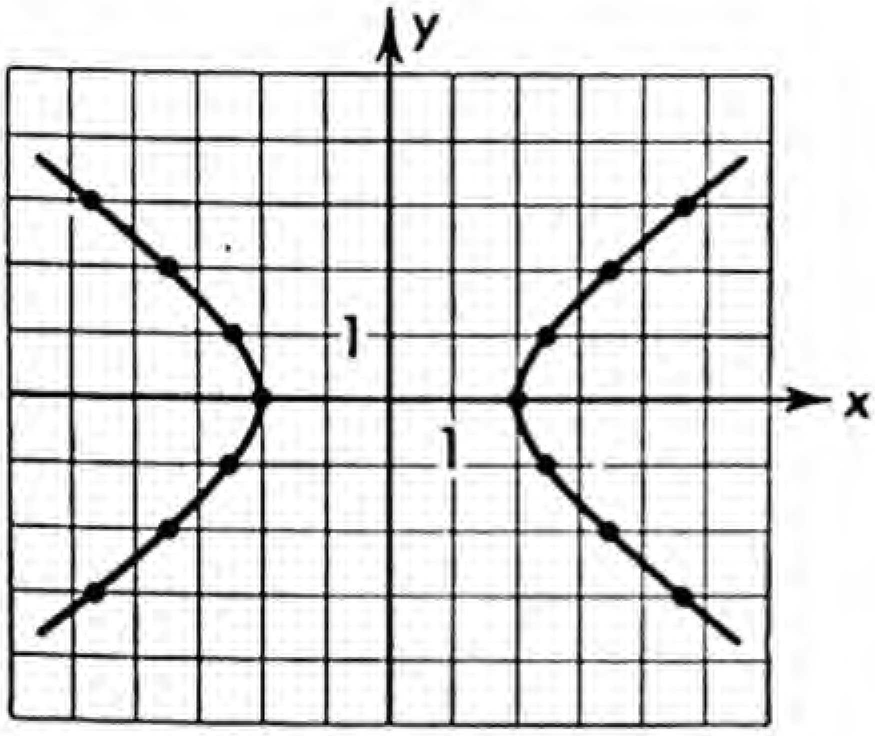


FIG. 18

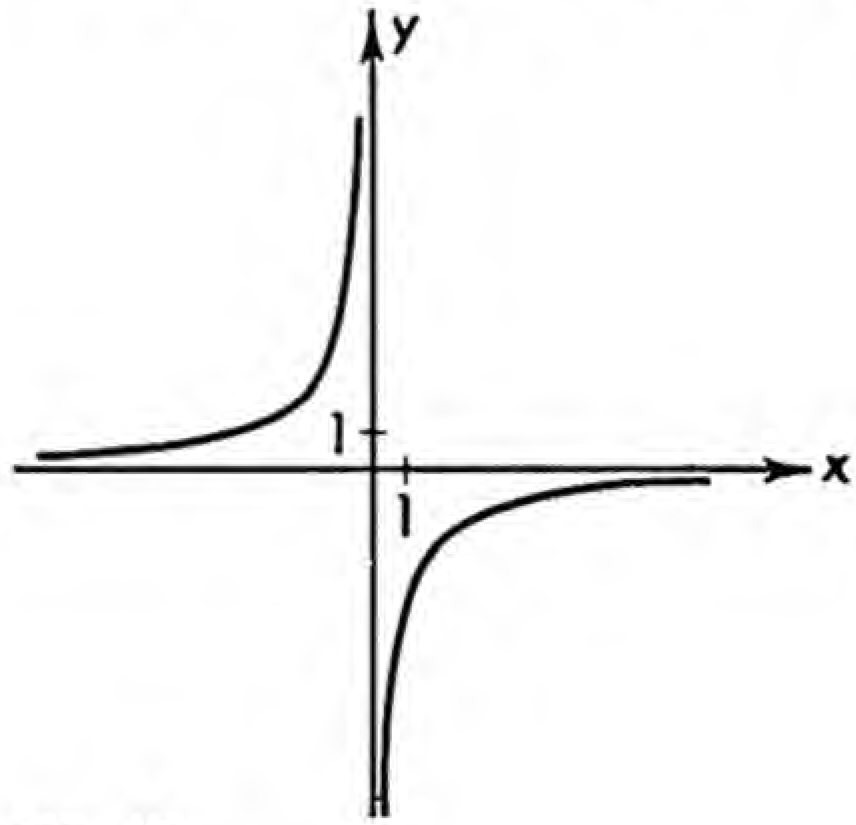


FIG. 19

Example 4. Graph the equation $xy = -4$.

Solution. Solve for y: $y = -\dfrac{4}{x}\cdot$ If $x = 0$, $y = -\dfrac{4}{0}\cdot$ Since division by zero is ruled out, we see that the value of y does not exist when $x = 0$, i.e., the curve does not meet the y-axis. The student should prepare a table of values and verify the **hyperbola** shown in Fig. 19.

Exercise 47

Solve graphically.

1. $\begin{cases} x^2 + y^2 = 25, \\ 4x + 3y = 25. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 9, \\ x - y = 5. \end{cases}$

3. $\begin{cases} y = x^2 - 2x - 5, \\ x + y = 1. \end{cases}$

4. $\begin{cases} 3x^2 + y^2 = 12, \\ 3x - 2y = 3. \end{cases}$

5. $\begin{cases} y^2 - x^2 = 7, \\ 2x - y = 1. \end{cases}$

6. $\begin{cases} xy = 6, \\ x - 2y = 4. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 20, \\ xy = 8. \end{cases}$

8. $\begin{cases} x^2 + 2y^2 = 43, \\ x^2 - y^2 = 16. \end{cases}$

9. $\begin{cases} 2x^2 + y^2 = 36, \\ y = 6 - x^2. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 16, \\ x = y^2 - 4. \end{cases}$

11. $\begin{cases} x^2 + 2y^2 = 16, \\ x^2 + y^2 = 36. \end{cases}$

12. $\begin{cases} y = x^2 - 4x, \\ y = -x^2 + 6. \end{cases}$

Solve algebraically.

13. $\begin{cases} x^2 + 2y^2 + x - 12 = 0, \\ x^2 - 3y^2 - 9x + 23 = 0. \end{cases}$

14. $\begin{cases} x^2 + y^2 - 8x - 8y = 20, \\ xy + 4x + 4y = 40. \end{cases}$

15. $\begin{cases} 2x^2 - xy + 5y^2 - x - 2y = 10, \\ 2x + 4y = 5. \end{cases}$

16. $\begin{cases} x^2 + xy + y^2 = 49, \\ x^2 - y^2 = 35. \end{cases}$

17. $\begin{cases} x + y + z = 4, \\ 3x - y + z = 2, \\ x^2 + y^2 + z^2 = 10. \end{cases}$

18. $\begin{cases} x^2 + 3y^2 - z^2 = 4, \\ x^2 + y^2 + z^2 = 14, \\ x + y = 3. \end{cases}$

19. $\begin{cases} x^3 + y^3 = 26, \\ x + y = 2. \end{cases}$

Hint. Divide the first equation by the second. Solve the resulting equation simultaneously with the linear equation.

20. For what values of the constant k will the line $y = 2x + k$ be tangent to the circle $x^2 + y^2 = 20$?

Hint. Eliminate y by substitution. The resulting equation in x must have equal roots, i.e., its discriminant must be zero.

21. Find the dimensions of the rectangle whose area is 205 square feet and whose perimeter is 92 feet.

22. One side of a rectangle is 3 feet longer than the side of a square. The area of the rectangle is twice that of the square. The perimeter of the rectangle exceeds that of the square by 10 feet. Find the dimensions of the two figures.

23. Each of two rectangles has an area of 288 square feet. One rectangle is 6 feet longer and 4 feet narrower than the other. Find their dimensions.

24. A rectangle is 12 feet long and 5 feet wide. Find the dimensions of a second rectangle whose area is twice that of the given rectangle and whose perimeter is twice the given rectangle's perimeter.

25. The altitude of an isosceles triangle is 24. Its perimeter is 64. Find the sides of the triangle.

26. Two machines together can perform a certain task in $2\frac{2}{5}$ hours. One machine working alone can perform the task in 2 hours less than the other. Find the time it takes each machine alone to perform the task.

27. The sum of the reciprocals of two numbers is $\frac{7}{12}$. The product of the numbers is 24. Find the numbers.

28. A sum of \$1000 is divided into two parts which draw simple interest at the same rate. At the end of two years one part amounts to \$528; at the end of four years the other part amounts to \$624. Find the rate of interest and how the original sum was divided.

29. In two hours a motorboat goes 9 miles downstream on a river and returns to its starting point. Its rate downstream is five times the rate of the current. Find the rate of the current and the speed of the boat in still water.

30. A speed boat requires $1\frac{1}{4}$ hours to go 20 miles upstream and return to its starting point. The same boat can go 15 miles upstream in $\frac{1}{2}$ hour. Find the rate of the current and the rate of the boat in still water.

31. A circular park is surrounded by a sidewalk which, in turn, is enclosed by a paved driveway whose width is four times that of the sidewalk. The area of the sidewalk is 2145π square feet; the area of the driveway is 9080π square feet. Find the width of the sidewalk and the radius of the park.

32. Two circles are tangent externally and have a combined area of 130π square feet. If the distance between their centers is 16 feet, find the radii of the two circles.

33. Two cities are 120 miles apart. Two autos start at the same time, one from each city, traveling toward the other city. They meet at the end of 1 hour and 12 minutes. The slower auto reaches its destination 1 hour after the faster auto does. Find the rates of the two autos.

34. Two airplanes leave simultaneously, one going from airport A to airport B, the other traveling from B to A. After they meet, the first plane requires $2\frac{1}{2}$ hours to reach B. The second plane arrives at A $3\frac{3}{5}$ hours after they meet. How long were they in the air before they met?

35. At noon a motorcycle traveling 60 mph passed an auto going in the same direction. At 2 P.M. the motorcycle met a second auto going in the opposite direction with the same speed as the first auto. The motorcycle was 35 miles away when the two autos met. Find the speed of the autos and the time at which they met.

36. An auto leaves a point B for a town 36 miles away. Five minutes later a motorcycle, sent to overtake the auto, leaves B with a speed of 60 mph. After overtaking the auto, the motorcycle returns to B and arrives at exactly the same time that the auto arrives in the town. Find the speed of the auto.

chapter 10 *Ratio, proportion, and variation*

82. Ratio. The **ratio** of a number a to another number b is the quotient $\frac{a}{b}$. The ratio of a to b is also written $a : b$. Every ratio is a fraction; any fraction may be considered as a ratio. The ratio of two concrete quantities has no meaning unless the quantities are of the same kind. For example, we do not speak of the ratio of 4 feet to 6 dollars, but the ratio of 4 feet to 6 feet is $\frac{4}{6}$ or $\frac{2}{3}$. Two like quantities must be expressed in terms of the same unit before we compute the ratio of one quantity to the other. Thus, the ratio of 9 inches to 2 feet is $\frac{9}{24}$ or $\frac{3}{8}$.

83. Proportion. A **proportion** is a statement that two ratios are equal. In the proportion *

$$\frac{a}{b} = \frac{c}{d} \tag{1}$$

the quantities b and c are called the **means;** a and d are called the **extremes.** The quantity d is called the **fourth proportional** to a, b, and c. If we clear of fractions in (1), we get $ad = bc$. This result is frequently stated as, "the product of the means is equal to the product of the extremes."

All quantities in a proportion need not be expressed in terms of the same unit. For example,

$$\frac{4 \text{ feet}}{6 \text{ feet}} = \frac{10 \text{ cents}}{15 \text{ cents}}.$$

* This is sometimes written $a : b = c : d$ and is read "a is to b as c is to d."

If $\frac{a}{x} = \frac{x}{b}$, then x is called a **mean proportional** between a and b. Clearing of fractions, we have $x^2 = ab$; $x = \pm\sqrt{ab}$. Hence a mean proportional between two numbers may be obtained be extracting the square root of their product.

If $\frac{a}{b} = \frac{b}{x}$, then x is called the **third proportional** to a and b.

Exercise 48

Express each ratio as a fraction and simplify.

1. $\frac{3}{10} : \frac{2}{5}$.
2. $\frac{7}{8} : \frac{1}{12}$.

Find the ratio of the given quantities.

3. 4 pounds to 6 ounces.
4. 12 quarts to 3 pints.
5. 120 feet to 1 mile.
6. 10 hours to 2 days.

Solve for x.

7. $\frac{x}{4} = \frac{5}{6}$.
8. $\frac{x+1}{x+2} = \frac{7}{9}$.
9. $\frac{3x-8}{x+2} = \frac{5x+4}{4x-1}$.
10. $\frac{6x+5}{4x+3} = 2$.

Find the mean proportional between each of the following pairs of numbers.

11. 2 and 32.
12. $\frac{1}{10}$ and 1000.
13. 4 and 9.
14. a^3 and a^7.

Find the fourth proportional to each of the following sets of numbers.

15. 2, 5, 6.
16. 3, 4, $\frac{1}{2}$.

If two polygons are similar, (a) their corresponding sides are proportional, and (b) their areas are proportional to the squares of any two corresponding sides.

17. A man $5\frac{3}{4}$ feet tall stands 30 feet from a lamppost. If the man's shadow is 10 feet long, how high is the light above the ground?

18. A flagpole casts a 52-foot shadow at the same time that a near-by 3-foot stake casts a shadow $4\frac{1}{3}$ feet long. How high is the flagpole?

19. The sides of a quadrilateral are 4, 5, 6, and 8 inches. The shortest side is decreased by 1 inch. By how much must the other sides be decreased to obtain a similar quadrilateral?

20. Two similar triangles have areas of 18 square feet and 50 square feet, respectively. The smaller triangle has a base 6 feet long. Find the base of the larger triangle.

21. A quadrilateral with sides 3, 4, 6, and 10 inches has an area of 12 square inches. Find the area of a similar quadrilateral whose longest side is 15 inches.

22. On a certain map, 3 inches represents 40 miles. Find the actual distance between two towns if the measured distance on the map is $5\frac{1}{2}$ inches.

Given the proportion

$$\frac{a}{b} = \frac{c}{d}, \tag{1}$$

derive the following proportions.

23. $\dfrac{b}{a} = \dfrac{d}{c}$. This proportion is said to be obtained from (1) by **inversion.**

24. $\dfrac{a}{c} = \dfrac{b}{d}$. This proportion is said to be obtained from (1) by **alternation.**

25. $\dfrac{a+b}{b} = \dfrac{c+d}{d}$. This proportion is said to be obtained from (1) by **composition.**

Hint. Add 1 to both sides of (1).

26. $\dfrac{a-b}{b} = \dfrac{c-d}{d}$. This proportion is said to be obtained from (1) by **division.**

Hint. Subtract 1 from both sides of (1).

27. $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$. This proportion is said to be obtained from (1) by **composition and division.**

Hint. Divide 25 by 26.

28. $\dfrac{a^3}{b^3} = \dfrac{c^3}{d^3}$.

29. $\dfrac{a+kb}{b} = \dfrac{c+kd}{d}$.

30. Given $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f}$, prove that $\dfrac{a+c+e}{b+d+f} = \dfrac{a}{b}$.

Hint. Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = k$. Then $a = kb$, $c = kd$, $e = kf$.

84. Direct variation. Let x and y be two variables and let k be any constant. *If*

$$y = kx,$$

we say that

y varies directly as x.

Other expressions having the same meaning are, "y varies as x,"* "y is directly proportional to x," and "y is proportional to x." The constant k is called the **constant of variation** or the factor of proportionality.

Illustration 1. The circumference of a circle varies directly as the radius because $c = 2\pi r$. In this case the constant of variation is 2π.

Illustration 2. The distance s traveled by a body falling from rest varies directly as the square of the time t of descent. This means that $s = kt^2$. The value of k depends upon the units used. If s is in feet and t in seconds, k is approximately 16. Then $s = 16t^2$.

85. Inverse variation. *If*

$$y = \frac{k}{x},$$

we say that

y varies inversely as x.

Illustration 1. Boyle's law states that at a fixed temperature, the volume of a gas varies inversely as the pressure: $v = \dfrac{k}{p}$.

86. Joint and combined variation. If $y = kxz$, we say that y varies **jointly** as x and z, i.e., y varies directly as x and directly as z, or y varies directly as the product of x and z.

Illustration 1. The area A of a triangle varies jointly as the altitude h and the base b because $A = \frac{1}{2}hb$.

The several types of variation may be combined.

Example 1. Given: y varies directly as the cube of x and inversely as the square of z.

(*a*) Write the equation of variation.
(*b*) How does y change if x is doubled?
(*c*) How does y change if z is doubled?

Solution.

(*a*) $$y = \frac{kx^3}{z^2}. \tag{1}$$

* The statement "y varies directly as x" is sometimes written $y \propto x$.

(*b*) If x is doubled, then x^3 becomes $(2x)^3$ or $8x^3$. Hence the right side of (1) is multiplied by 8. Therefore, if x is doubled, y is multiplied by 8.

(*c*) If z is doubled, z^2 is multiplied by 4, and y is multiplied by $\frac{1}{4}$.

Example 2. The safe load of a beam supported at both ends varies jointly as the breadth and the square of the depth and inversely as the length between the supports. A pine beam 2 inches wide, 6 inches deep, and 12 feet long can safely support a weight of 400 pounds.

(*a*) Find the safe load of a beam of the same material 4 inches wide, 10 inches deep, and 16 feet long.

(*b*) Find the depth needed for a pine beam 3 inches wide and 20 feet long if it is to support a load of 640 pounds.

Solution. Write the general equation of variation:

$$L = \frac{kbd^2}{l}.$$

Determine k by setting $L = 400$, $b = 2$, $d = 6$, $l = 12$:

$$400 = \frac{k(2)(6^2)}{12}; \quad k = \frac{200}{3}.$$

Write the equation of variation for pine beams:

$$L = \frac{200}{3} \cdot \frac{bd^2}{l}. \tag{1}$$

This equation, or formula, can be used only if L is in pounds, b is in inches, d is in inches, and l is in feet.

(*a*) Substitute the data in (1):

$$L = \frac{200}{3} \cdot \frac{4(10^2)}{16} = 1666\tfrac{2}{3} \text{ pounds.}$$

(*b*) Substitute in (1):

$$640 = \frac{200}{3} \cdot \frac{3d^2}{20}; \quad d^2 = 64; \quad d = 8 \text{ inches.}$$

Exercise 49

Translate each statement into an equation and then insert the value of the constant of variation.

1. The area A of a circle varies directly as the square of its radius r.

2. The area A of a cube varies directly as the square of its edge E.

3. The area A of a trapezoid varies jointly as its altitude h and the sum of its bases, b_1 and b_2.

4. The volume V of a cylinder varies jointly as its altitude h and the square of its radius r.

Write the equation of variation. Determine the constant of variation from the given data. Rewrite the equation, inserting the value of the constant of variation.

5. W varies directly as x and inversely as d^2; $W = 6$ when $x = 4$ and $d = 5$.

6. U varies jointly as x and $\sqrt{t}$; $U = 8$ when $x = 5$ and $t = 9$.

7. R varies inversely as s and t^2; $R = \frac{2}{3}$ when $s = 6$ and $t = 2$.

8. S varies directly as u^3 and inversely as w and z^2; $S = 20$ when $u = 2$, $w = 4$, and $z = 3$.

9. If y varies directly as x^2 and $\sqrt{t}$ and inversely as z^3, write the equation of variation. How does y change (*a*) if x is doubled, (*b*) if t is doubled, (*c*) if z is doubled, (*d*) if x, t, and z are doubled?

10. If M varies directly as r and inversely as s and t^2, write the equation of variation. How does M change (*a*) if r is halved, (*b*) if s is halved, (*c*) if t is halved, (*d*) if r and t are halved and s is doubled?

11. If y varies directly as x^3, and if $y = 40$ when $x = 2$, find the value of x when $y = 135$.

12. If F varies inversely as l^2, and if $F = 90$ when $l = 3$, find the value of l when $F = 10$.

13. If w varies directly as x^2 and inversely as t, and if $w = 60$ when $x = 3$ and $t = 10$, find the value of w when $x = 4$ and $t = 8$.

14. If z varies jointly as r and s^2 and inversely as t^3, and if $z = 96$ when $r = 4$, $s = 3$, and $t = 2$, find the value of z when $r = 6$, $s = 8$, and $t = 4$.

15. If M varies directly as x and inversely as y^2 and z, and if $M = 100$ when $x = 3$, $y = 2$, and $z = 7$, find the value of z when $M = 200$, $x = 4$, and $y = 1$.

16. If y varies directly as 2^x and if $y = 40$ when $x = 3$, find the value of y when $x = 6$.

17. The cross-sectional area of a properly designed chimney varies directly as the amount of coal used per hour and inversely as the square root of the height of the chimney. A chimney 100 feet high connected with a furnace using 5 tons of coal per hour has a cross-sectional area of 30 square feet. Find the proper cross-sectional area for an 81-foot chimney connected with a furnace using 3 tons of coal per hour.

18. The horsepower required to drive an airplane varies directly as the cube of the speed. If a plane uses 240 horsepower when traveling 200 mph, find the horsepower needed to drive the plane 300 mph.

19. The horsepower that a shaft can safely transmit varies jointly as its speed and the cube of its diameter. A 4-inch shaft making 120 rpm can transmit 96 horsepower. Find the horsepower that can be transmitted by a 3-inch shaft of the same material when it makes 160 rpm.

20. The altitude of a cone varies directly as its volume, and inversely as the square of the radius of its base. If a cone with altitude 5 and radius 6 has a volume of 60π, find the altitude of a cone having a radius of 9 and a volume of 54π.

21. The weight of a body on the surface of a planet varies directly as the planet's mass and inversely as the square of the planet's radius. If a co-ed weighs 100 pounds on Earth, what would she weigh on Mars, whose mass is $\frac{1}{9}$ that of Earth and whose radius is 2000 miles? (Assume Earth's mass as 1 and Earth's radius as 4000 miles.)

22. The electrical resistance of a wire of given composition varies directly as its length and inversely as the square of its diameter. A copper wire 90 feet long with a diameter of 0.06 inch has a resistance of 0.26 ohm. Find the length of a copper wire with a diameter of 0.08 inch if it has a resistance of 0.39 ohm.

23. The volume of a gas varies directly as its absolute temperature and inversely as the pressure to which the gas is subjected. A certain gas confined in a cylinder occupies a volume of 200 cubic inches when the absolute temperature is 300° and the pressure is 40 pounds per square inch. Find the pressure needed to reduce the volume of the gas to 160 cubic inches when the temperature is 312°.

24. The time of oscillation of a pendulum varies directly as the square root of its length. If a pendulum 36 inches long makes one oscillation in 0.96 second, how long is a pendulum that makes one oscillation in 0.48 second?

25. "If a hen and a half lays an egg and a half in a day and a half," how long will it take 6 hens to lay 12 eggs?

26. The number of horsepower developed by an engine varies jointly as its piston displacement and its speed. An engine with a piston displacement of 320 cubic inches develops 144 horsepower at a speed of 3600 rpm. Find

the horsepower developed by an engine with a piston displacement of 280 cubic inches when its speed is 3000 rpm.

27. The safe load for a circular wooden pillar varies directly as the fourth power of its diameter and inversely as the square of its length. A 6-in. pillar 12 feet long can safely support a load of 5.4 tons. Find the proper diameter for an 8-foot pillar made of the same material if it must support a weight of 2.4 tons.

28. The force of the wind against a perpendicularly placed flat surface varies directly as the surface area and the square of the wind velocity. The force on a sail having an area of 10 square feet is 6 pounds when the wind blows 12 mph. Find the force on 8 square feet of sail when the wind velocity is 15 mph.

29. The amount of heat developed in an electric circuit varies jointly as the resistance, the time of flow of the current, and the square of the current. A toaster using a current of 3 amperes and having a resistance of 40 ohms produces 1728 gram-calories in 20 seconds. How long will it take a flatiron using a current of 5 amperes and having a resistance of 24 ohms to produce the same amount of heat?

30. Kepler's third law of planetary motion says that the square of the time required for a planet to revolve about the sun varies directly as the cube of its distance from the sun. The distance from Neptune to the sun is 30 times the distance from the Earth to the sun. How many years does it take Neptune to revolve about the sun?

Translate each equation into a statement in the language of variation.

31. $A = \dfrac{3Cr^2}{\sqrt{l}}$.

32. $y = \dfrac{2r^3}{7st^2}$.

33. If r varies inversely as s, and s varies inversely as t, show that r varies directly as t.

34. If v varies directly as w, and w varies directly as x, show that v varies directly as x.

35. If y varies directly as x, and if $y = 6$ when $x = 2$, graph the equation of variation.

36. If y varies inversely as x, and if $y = 2$ when $x = 4$, graph the equation of variation.

chapter 11 Progressions

87. Arithmetic progressions. If a set of numbers is arranged in a definite order, it is called a **sequence.** The numbers in the sequence are called **terms.** An **arithmetic progression** is a sequence of numbers each of which (after the first) is obtained from the preceding number by adding a fixed quantity called the **common difference.** For example, the sequence 1, 4, 7, 10 is an arithmetic progression in which the common difference is 3. The common difference can be found by subtracting any term from the following one.

88. The *n*th term of an arithmetic progression. Let a be the 1st term and let d be the common difference. Then the 2nd term is $a + d$; the 3rd term is $a + 2d$; the 4th term is $a + 3d$. In each term the coefficient of d is one less than the number of the term. Let n represent the number of terms and let l designate the nth term. Then

$$l = a + (n - 1)d. \quad [1]$$

Illustration. In the A.P.* 7, 3, $-1, \cdots$, we have $a = 7$, $d = -4$. The 20th term is $l = 7 + 19(-4) = 7 - 76 = -69$.

In the progression 7, 3, $-1, \cdots$, the three dots (to be read "and so on") indicate that the sequence is to be continued indefinitely.

89. The sum of the terms of an arithmetic progression. Let S represent the sum of the n terms of an A.P. If we write the terms in natural order and then in reverse order, we get

* Arithmetic progression.

$$S = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l,$$
$$S = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a.$$

Add the two equations:

$$2S = (a + l) + (a + l) + (a + l) + \cdots \text{to } n \text{ terms} = n(a + l).$$

$$S = \frac{n}{2}(a + l). \qquad [2]$$

Example 1. Find the sum of 20 terms of the A.P. $5, 8, 11, \cdots$.

Solution. We know $a = 5$, $d = 3$, $n = 20$. Using [1], we get $l = 5 + 19 \cdot 3 = 62$. Then using [2], we obtain $S = \frac{20}{2}(5 + 62) = 670$.

If we know any three of the quantities l, a, n, d, S, we can find the remaining two quantities by using formulas [1] and [2], either successively or simultaneously. The value of n must always be a positive integer. Why?

Example 2. In a certain A.P., $d = 2$, $l = 11$, $S = 35$. Find a and n.

Solution. Substitute in [1]:

$$11 = a + (n - 1)2.$$

Substitute in [2]:

$$35 = \frac{n}{2}(a + 11).$$

To solve these two equations simultaneously, solve the first for a: $a = 13 - 2n$. Then substitute in the second equation:

$$35 = \frac{n}{2}(13 - 2n + 11).$$
$$n^2 - 12n + 35 = 0; \quad n = 5, 7.$$

For $n = 5$, $a = 13 - 2n = 13 - 10 = 3$.
For $n = 7$, $a = 13 - 14 = -1$.

The two progressions are

$$3,\ 5,\ 7,\ 9,\ 11$$

and

$$-1,\ 1,\ 3,\ 5,\ 7,\ 9,\ 11.$$

If we eliminate l from [1] and [2], we get

$$S = \frac{n}{2}[2a + (n - 1)d]. \qquad [3]$$

This formula is convenient but not necessary.

90. Arithmetic means. The terms of an A.P. that lie between two given terms are the **arithmetic means** between these terms. Thus, in the A.P. 3, 7, 11, 15, the arithmetic means between 3 and 15 are 7 and 11.

Example 1. Insert three arithmetic means between 7 and 10.

Solution. We must form an A.P. having $a = 7$, $l = 10$, and $n = 5$ (since there are three means and the two given terms). Use formula [1]: $10 = 7 + 4d$; $d = \frac{3}{4}$. The progression is

$$7, \quad 7\tfrac{3}{4}, \quad 8\tfrac{1}{2}, \quad 9\tfrac{1}{4}, \quad 10.$$

The required means are $7\frac{3}{4}$, $8\frac{1}{2}$, $9\frac{1}{4}$.

If a single arithmetic mean is inserted between two numbers, it is called **the arithmetic mean** of the numbers. Let x be the arithmetic mean of the numbers a and b. Then a, x, b, is an A.P. Recall that the common difference may be obtained by subtracting any term from the following one. Hence $x - a = b - x$, or $x = \dfrac{a + b}{2}$. Therefore the arithmetic mean of two numbers is one-half of their sum. This is commonly called the **average** of the two numbers.

Exercise 50

Which of the following sequences are arithmetic progressions? Continue each A.P. to two additional terms.

1. $-5, -1, 3.$

2. $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}.$

3. 36, 25, 16.

4. $3x - 8y, 2x - 4y, x.$

Use formulas to find l and S for each of the following progressions.

5. 3, 10, 17, $\cdots$ to 30 terms.

6. 1.05, 1.10, 1.15, $\cdots$ to 13 terms.

7. 6, 5.97, 5.94, $\cdots$ to 10 terms.

8. $7, 3, -1, \cdots$ to 12 terms.

9. $10, 8\frac{1}{3}, 6\frac{2}{3}, \cdots$ to 16 terms.

10. $2, \frac{17}{6}, \frac{11}{3}, \cdots$ to 11 terms.

Use formulas to find the required quantities for each arithmetic progression.

11. Given $d = 6$, $n = 51$, $l = 319$; find a and S.

12. Given $d = -2$, $n = 42$, $l = 0$; find a and S.

13. Given $n = 5$, $l = 3$, $S = 16$; find a and d.

14. Given $a = -3$, $n = 10$, $S = 330$; find d and l.

15. Given $a = 37$, $n = 6$, $S = 117$; find d and l.

16. Given $a = 5$, $l = 7$, $S = 54$; find d and n.

17. Given $a = 3$, $d = 2$, $S = 195$; find n and l.

18. Given $a = 2$, $d = -\frac{1}{3}$, $S = 6$; find n and l.

19. Given $d = 3$, $n = 7$, $S = 28$; find a and l.

20. Given $d = -6$, $n = 8$, $S = 0$; find a and l.

21. Given $d = \frac{1}{2}$, $l = 3$, $S = 10$; find a and n.

22. Given $d = 4$, $l = 23$, $S = 33$; find a and n.

23. Given $a = 382$, $d = 5$, $l = 777$; find n and S.

24. How many terms are in the A.P. $1.8, 2.5, 3.2, \cdots, 39.6$?

25. Insert three arithmetic means between 5 and 10.

26. Insert five arithmetic means between 7 and 9.

27. Insert four arithmetic means between 3 and 12.

28. Insert two arithmetic means between -2 and 10.

29. Find the arithmetic mean of 1492 and 2060.

30. Find the arithmetic mean of 7806 and 8541.

31. Find the sum of all even integers from 136 to 544 inclusive.

32. Show that the sum of the first n positive integers is $\frac{1}{2}n(n + 1)$.

33. Show that the sum of the first n positive odd integers is n^2.

34. Find the value of x for which $7, x + 8, 5x$ is an A.P.

35. Derive a formula for S in terms of d, n, l.

36. Derive a formula for S in terms of a, d, l.

37. Find the 1st term of an A.P. in which the 23rd term is 444 and the 91st term is 478.

Hint. Consider 444 as the 1st term of another A.P.; then find d.

38. Find the sum of the first 10 terms of an A.P. in which the 3rd term is 9 and the 7th term is 25.

39. In rolling down a hill, a barrel travels $1\frac{1}{2}$ feet in the 1st second, $4\frac{1}{2}$ feet in the 2nd second, $7\frac{1}{2}$ feet in the 3rd second, and so on. How far does it travel in 20 seconds?

40. Thirty-six logs are to be piled in layers, the top layer to consist of 1 log, the next layer 2 logs, the 3rd layer 3 logs, and so on. How many logs should be placed in the bottom layer?

41. A man saved a total of \$8250 in 10 years. In each year, after the first, he saved \$50 more than he did in the preceding year. How much did he save in the first year?

42. Find the sum of all multiples of 11 between 100 and 1000.

91. Geometric progressions. A **geometric progression** is a sequence of numbers each of which (after the first) is obtained by multiplying the preceding number by a fixed quantity called the **common ratio.** For example, the sequence 2, 6, 18, 54 is a geometric progression in which the common ratio is 3. The common ratio can be found by dividing any term by the *preceding* one.

92. The nth term of a geometric progression. Let a be the 1st term and let r be the common ratio. Then the 2nd term is ar; the 3rd term is ar^2; the 4th term is ar^3. In each term the exponent of r is one less than the number of the term. Let n represent the number of terms and let l designate the nth term. Then

$$\boldsymbol{l = ar^{n-1}}. \tag{1}$$

Illustration. In the G.P.* 5, -10, 20, $\cdots$, we have $a = 5$, $r = -2$. The 8th term is $l = 5(-2)^7 = 5(-128) = -640$.

93. The sum of the terms of a geometric progression. Let S represent the sum of the n terms of a G.P. Then

$$S = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1}. \qquad (1)$$

Multiply both sides by r:

$$rS = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \qquad (2)$$

Subtract (2) from (1):

$$S - rS = a - ar^n.$$

$$S(1 - r) = a - ar^n.$$

$$\boldsymbol{S = \frac{a - ar^n}{1 - r}}. \tag{2}$$

* Geometric progression.

Since $ar^n = r(ar^{n-1}) = rl$, we may rewrite [2],

$$S = \frac{a - rl}{1 - r}. \qquad [3]$$

Example 1. Find the sum of six terms of the G.P. 8, 16, 32, $\cdots$.

Solution. We know $a = 8$, $r = 2$, $n = 6$. Using [2], we get

$$S = \frac{8 - 8(2)^6}{1 - 2} = \frac{8 - 512}{-1} = 504.$$

Example 2. In a certain G.P., $a = 96$, $l = 3$, $S = 189$. Find r and n.

Solution. Substitute in [3] to find r:

$$189 = \frac{96 - 3r}{1 - r}.$$

$$189 - 189r = 96 - 3r; \quad 93 = 186r; \quad r = \tfrac{1}{2}.$$

Then substitute in [1] to find n:

$$3 = 96(\tfrac{1}{2})^{n-1}; \quad (\tfrac{1}{2})^{n-1} = \tfrac{3}{96} = \tfrac{1}{32} = (\tfrac{1}{2})^5.$$

Hence $n - 1 = 5$; $n = 6$.

The progression is

$$96, \quad 48, \quad 24, \quad 12, \quad 6, \quad 3.$$

If we know any three of the quantities S, n, a, r, l, we can find the remaining two quantities by using formulas [1], [2], [3], with the restriction that n must be a positive integer.

94. Geometric means. The terms of a G.P. that lie between two given terms are called **geometric means** between these terms.

Example 1. Insert two geometric means between 16 and 54.

Solution. We must form a G.P. with $a = 16$, $l = 54$, and $n = 4$. Use formula [1]: $54 = 16r^3$; $r^3 = \frac{54}{16} = \frac{27}{8}$; $r = \frac{3}{2}$. The progression is

$$16, \quad 24, \quad 36, \quad 54.$$

If a single geometric mean is inserted between two numbers, it is called **a geometric mean** of the numbers. The student should show that if a, x, b is a G.P., then $x = \pm\sqrt{ab}$. Hence a geometric mean of two numbers may be obtained by extracting the square root of their product.*

* A geometric mean of two numbers is the same as a *mean proportional* between the numbers (Art. 83).

Exercise 51

Which of the following sequences are geometric progressions? Continue each G.P. to two additional terms.

1. $4, -20, 100.$ **2.** $10, -5, 1.$

3. $3, 3\sqrt{2}, 6.$ **4.** $2, .2, .02.$

5. $81, 54, 36.$ **6.** $\frac{1}{6}, \frac{1}{4}, \frac{3}{8}.$

7. $(1.05)^2, (1.05)^4, (1.05)^8.$ **8.** $9, 3, 0.$

Use formulas to find l and S for each of the following progressions.

9. $3, 6, 12, \cdots$ to 7 terms. **10.** $1024, 512, 256, \cdots$ to 11 terms.

11. $7, -21, 63, \cdots$ to 6 terms. **12.** $\frac{4}{3}, 4, 12, \cdots$ to 5 terms.

13. $768, -384, 192, \cdots$ to 8 terms. **14.** $8, -16, 32, \cdots$ to 8 terms.

15. $7, 70, 700, \cdots$ to 6 terms. **16.** $3, -.3, .03, \cdots$ to 7 terms.

17. $\frac{1}{9}, \frac{1}{6}, \frac{1}{4}, \cdots$ to 5 terms.

18. $1, (1.03), (1.03)^2, \cdots$ to 40 terms.

Hint. Find from Table IV that $(1.03)^{39} = 3.1670.$

Use formulas to find the required quantities for each geometric progression.

19. Given $r = 6, n = 4, l = 108$; find a and S.

20. Given $r = -\frac{1}{2}, n = 7, l = \frac{3}{4}$; find a and S.

21. Given $r = -2, n = 5, S = 33$; find a and l.

22. Given $r = 3, n = 4, S = 80$; find a and l.

23. Given $a = 2, r = -3, S = 122$; find n and l.

24. Given $a = 5, r = 2, S = 635$; find n and l.

25. Given $a = 4, l = 324, S = 484$; find r and n.

26. Given $a = 6, l = -384, S = -306$; find r and n.

27. Given $r = \frac{1}{2}, l = 1, S = 127$; find a and n.

28. Given $r = -3, l = 972, S = 732$; find a and n.

29. Insert two geometric means between 5 and 10.

30. Insert four geometric means between 96 and -3.

31. Insert two different sets of five geometric means between $\frac{1}{4}$ and 16.

32. Insert two different sets of three geometric means between $\frac{1}{16}$ and 10,000.

33. Find a geometric mean of 16 and 25.

34. Find a geometric mean of $\frac{1}{9}$ and 64.

35. Derive a formula for S in terms of r, l, n.

36. Find the 7th term of a G.P. in which the 2nd term is -486 and the 5th term is 144.

37. For what values of x does $x - 4$, $x - 2$, $2x - 1$ form a G.P.?

38. In a certain G.P., $a = 1$, $n = 3$, $S = 91$. Use formulas to find r and l. (Two solutions.)

Hint. $\dfrac{a - ar^3}{1 - r} = a(1 + r + r^2)$.

39. In a certain tournament there are three prizes totaling \$61. The 1st prize is 25% larger than the 2nd prize which is 25% larger than the 3rd prize. Find the values of the three prizes.

40. Given $3^8 = 6561$, find the sum of the first 15 positive integral powers of 3.

41. A 200-gallon cask is filled with alcohol. Twenty gallons are drawn off and replaced with water. Then 20 gallons of the mixture are drawn off and replaced with water. This continues until 4 drawings and replacements have been made. How much alcohol remains in the final mixture?

42. The midpoints of a square are connected to form a 2nd square, whose midpoints are connected to form a 3rd square, and so on. Find the perimeter of the 7th square if the first square has an area of 64 square inches.

95. Infinite geometric progressions. A geometric progression in which the number of terms increases without limit is called an **infinite geometric progression.**

Let A and B be two points that are two miles apart (Fig. 20). Suppose that an automobile starts at A and travels half the distance to B in 1 minute. Assume that in the 2nd minute it goes half as far as it did in the 1st minute. Suppose that in the 3rd minute it travels half the distance it covered in the 2nd minute. If this continues indefinitely, the distances traveled by the automobile, minute by minute, form the following infinite G.P.:

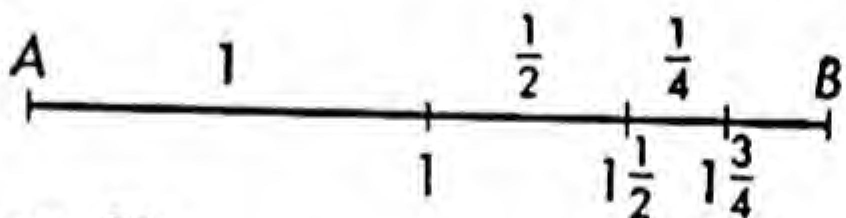


FIG. 20

$$1, \ \tfrac{1}{2}, \ \tfrac{1}{4}, \ \tfrac{1}{8}, \ \cdots.$$

Although the car is always approaching B, it will never reach it. We can say, however, that the automobile will, in time, reach any designated point that is between A and B, regardless of how close

the point is to B. For example, the point which is only $\frac{1}{1000}$ mile from B is reached and passed by the car in the 11th minute because, using $S = \dfrac{a - ar^n}{1 - r}$ with $a = 1, r = \frac{1}{2}, n = 11$, we have

$$S = \frac{1 - 1(\frac{1}{2})^{11}}{\frac{1}{2}} = \frac{1}{\frac{1}{2}} - \frac{\frac{1}{2048}}{\frac{1}{2}} = 2 - \frac{1}{1024} = 1\tfrac{1023}{1024}.$$

Hence at the end of the 11th minute the auto is only $\frac{1}{1024}$ mile from B. Since we can make the sum of the infinite G.P. come as close to 2 as we please, we say that the limit of its sum is 2. This is written $\lim\limits_{n \to \infty} S_n = 2$, and is read, "the limit of the sum of n terms as n becomes infinite is 2."

In the general infinite G.P.

$$a, \quad ar, \quad ar^2, \quad \cdots,$$

if $|r| < 1$,* we can make r^n come as close to 0 as we please by taking n sufficiently large. This is written $\lim\limits_{n \to \infty} r^n = 0$. We have then

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a - ar^n}{1 - r} = \frac{a - a \cdot 0}{1 - r} = \frac{a}{1 - r}.$$

For brevity we write:

If $|r| < 1$ *in a G.P.,*

$$S_\infty = \frac{a}{1 - r}. \qquad [4]$$

Example 1. Find the limit of the sum of the infinite G.P.

$$2, \tfrac{2}{5}, \tfrac{2}{25}, \cdots.$$

Solution. Use [4] with $a = 2$ and $r = \frac{1}{5}$:

$$S_\infty = \frac{2}{1 - \frac{1}{5}} = \frac{2}{\frac{4}{5}} = \frac{5}{2} = 2\tfrac{1}{2}.$$

96. Repeating decimals. If a real number can be expressed as the quotient of two integers, it is called a **rational number.** A repeating decimal is one in which the figures repeat themselves after a certain

* $|r| < 1$ is read "the absolute value (Art. 5) of r is less than 1." It means that r is between -1 and 1: $-1 < r < 1$.

point. All repeating decimals are rational numbers. For example, the repeating decimal 2.4135135 · · · (sometimes written $2.4\dot{1}3\dot{5}$) * is equal to $\frac{893}{370}$. The student should check this by division.

Example 1. Express the following repeating decimal as the quotient of two integers: 1.4272727 · · ·.

Solution.

$$1.4272727 \cdots = 1.4 + (.027 + .00027 + .0000027 + \cdots).$$

The terms in parentheses form an infinite G.P. in which $a \doteq .027$ and $r = .01$. For these terms

$$S_\infty = \frac{.027}{1 - .01} = \frac{.027}{.99} = \frac{27}{990} = \frac{3}{110}.$$

The given number $= \frac{14}{10} + \frac{3}{110} = \frac{157}{110}$.

It can be shown that if one integer is divided by another integer, the result is either an ending decimal or a repeating decimal. For example, $\frac{3}{8} = .375$, whereas $\frac{1}{7} = .\dot{1}4285\dot{7}$. Nonending, nonrepeating decimals form the class of irrational numbers.

Exercise 52

Find the limit of the sum of each infinite G.P.

1. $6, 2, \frac{2}{3}, \cdots$.
2. $100, 20, 4, \cdots$.
3. $\frac{1}{2}, \frac{1}{12}, \frac{1}{72}, \cdots$.
4. $9, -3, 1, \cdots$.
5. $.9, -.09, .009, \cdots$.
6. $2, \sqrt{2}, 1, \cdots$.

Express each repeating decimal as the quotient of two integers.

7. .222 · · ·.
8. .777 · · ·.
9. .1818 · · ·.
10. .4545 · · ·.
11. 3.4242 · · ·.
12. 2.5151 · · ·.
13. 1.23636 · · ·.
14. 5.67272 · · ·.
15. .495495 · · ·.
16. .801801 · · ·.
17. 1.234234 · · ·.
18. $.\dot{2}8571\dot{4} \cdots$.

19. The limit of the sum of an infinite G.P. is 20. Find the first term if the ratio is $\frac{1}{5}$.

20. The limit of the sum of an infinite G.P. is 40. Find the common ratio if the first term is 44.

* The dots over the figures indicate that these figures together with those between them constitute the repeating part of the decimal.

21. A ball is dropped from a height of 20 feet. Each time it strikes the ground it rebounds $\frac{3}{4}$ of the height from which it last fell. How far will it travel before coming to rest?

22. A wheel is making 90 revolutions per second. In each subsequent second it makes $\frac{2}{3}$ as many revolutions as in the preceding second. How many revolutions will it make before stopping?

97. Harmonic progressions. A **harmonic progression** is a sequence of numbers whose reciprocals form an arithmetic progression. Thus, $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}$ is a harmonic progression because 2, 5, 8, 11 is an A.P.

Example 1. Find the 9th term of the harmonic progression 3, 2, $\frac{3}{2}, \cdots$.

Solution. The reciprocals of these terms form the A.P. $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \cdots$, in which $d = \frac{1}{6}$. The 9th term of this A.P. is $l = \frac{1}{3} + 8(\frac{1}{6}) = \frac{5}{3}$. Hence the 9th term of the given harmonic progression is $\frac{3}{5}$.

The terms of a harmonic progression that lie between two given terms are the **harmonic means** between these terms. If a single harmonic mean is inserted between two numbers, it is called **the harmonic mean** of the numbers.

Example 2. Insert three harmonic means between $\frac{1}{10}$ and $\frac{1}{42}$.

Solution. We shall first insert three arithmetic means between 10 and 42. Using $l = a + (n - 1)d$, with $a = 10$, $l = 42$, $n = 5$, we get $42 = 10 + 4d$; $d = 8$.

The arithmetic progression is 10, 18, 26, 34, 42.

The harmonic progression is $\frac{1}{10}, \frac{1}{18}, \frac{1}{26}, \frac{1}{34}, \frac{1}{42}$.

Exercise 53

Which of the following sequences are harmonic progressions? Continue each harmonic progression to two additional terms.

1. $\frac{1}{3}, \frac{2}{5}, \frac{1}{2}$. **2.** $\frac{1}{23}, \frac{1}{17}, \frac{1}{11}$. **3.** 12, 6, 4. **4.** 9, 6, 4.

5. Find the 20th term of the harmonic progression $\frac{1}{7}, \frac{1}{11}, \frac{1}{15}, \cdots$.

6. Find the 12th term of the harmonic progression $\frac{3}{2}, \frac{4}{3}, \frac{6}{5}, \cdots$.

7. Insert five harmonic means between $\frac{1}{7}$ and $\frac{1}{19}$.

8. Insert two harmonic means between 1 and $\frac{1}{3}$.

9. Insert three harmonic means between $\frac{9}{26}$ and $\frac{3}{4}$.

10. Find the harmonic mean of 3 and $\frac{6}{7}$.

11. Derive a formula for the harmonic mean of a and b.

12. Find the 1st term of a harmonic progression whose 7th term is $\frac{2}{11}$ and whose 10th term is $\frac{1}{8}$.

13. A plane flies from A to B with a speed of x miles per hour. Returning from B to A its speed is y miles per hour. Show that its average speed for the round trip is the harmonic mean of x and y.

14. Find the value of x for which x, $x + 2$, $x + 5$ is a harmonic progression.

Exercise 54 (Miscellaneous Problems)

1. Find the next term of the progression 6, 8, 12, $\cdots$.

2. Find the next term of the progression $\frac{1}{30}$, $\frac{1}{20}$, $\frac{1}{15}$, $\cdots$.

3. Find the next term of the progression 64, 48, 36, $\cdots$.

4. Find the next term of the progression 20, 24, 30, $\cdots$.

5. Find the 50th term and the sum of the first 50 terms of the progression -3, 7, 17, $\cdots$.

6. Find the 6th term and the sum of the first 6 terms of the progression -15, 30, -60, $\cdots$.

7. Find the 10th term of the progression $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{8}$, $\cdots$.

8. Find the 20th term and the sum of the first 20 terms of the progression 95, 84, 73, $\cdots$.

9. Find the 5th term and the sum of the first 5 terms of the progression 100, 30, 9, $\cdots$.

10. Find the 5th term of the progression 3, $1\frac{1}{3}$, $\frac{6}{7}$, $\cdots$.

11. A pile of posts consists of 16 layers. The top layer contains 5 posts; the 2nd layer has 6 posts; the 3rd layer has 7 posts, and so on. How many posts are in the pile?

12. In a certain lottery the tickets are numbered 1, 2, 3, and so on, and are drawn at random from a hat. The price of each ticket in cents is the same as the number marked on it. How much money is taken in if 40 tickets are sold?

13. How many ancestors (2 parents, 4 grandparents, etc.) does a person have in the preceding 12 generations if each ancestor appears in only one line of descent?

14. Each stroke of an air pump removes $\frac{7}{10}$ of the air in a container. What fractional part of the air remains after 5 strokes?

15. Bread dough in a cup is rising fast enough to double its volume every minute. If the cup will be full at the end of 16 minutes, how many minutes will be needed before the cup is half full?

16. A ball is dropped from a height of 9 feet. Each time it strikes the ground it rebounds $\frac{2}{3}$ of the height from which it last fell. (*a*) How far has it traveled when it hits the ground for the fifth time? (*b*) How far will it travel before coming to rest?

17. Twenty potatoes are placed in a straight line on the ground at intervals of 3 feet. A basket is in line with the potatoes and is 10 feet from the nearest one. What is the total distance traveled by a contestant in the potato race if he must start at the basket and gather, one by one, the 20 potatoes?

18. In a certain high school, a teacher begins with a yearly salary of $1700 and receives an increase of $75 each succeeding year. (*a*) How much does a teacher make in his 12th year? (*b*) Find his total income in the first 12 years.

19. Find the sum of the first 40 positive integral multiples of 7.

20. In rolling down an inclined plane, a ball travels 5 inches during the 1st second. In each succeeding second, it travels 10 inches farther than in the preceding second. If the inclined plane is 60 feet long, how many seconds does it take the ball to reach the bottom?

21. On its first swing, a pendulum bob travels 10 inches. On each succeeding swing, it travels $\frac{4}{5}$ as far as on the preceding swing. (*a*) How far does it travel in five swings? (b) How far does it travel before it stops?

22. Find a G.P. of 3 terms such that the sum of the first 2 terms is 10, and the third term exceeds the second by 15. (Two solutions.)

23. Given $(1.03)^{46} = 3.895$, find the sum of the first 45 positive integral powers of 1.03.

24. The reciprocals of the terms of an A.P. form a harmonic progression. What kind of a sequence is formed by the reciprocals of a G.P.?

25. A man secures a loan of $5000. He promises to pay $500 of the principal at the end of each year until the debt is discharged. He also promises to pay, at the end of each year, 4% interest on the principal outstanding during that year. Find the total amount he pays in discharging his debt.

26. A man bequeathed his estate of $24,000 to his six children in such a way that the eldest received $400 less than the second eldest, who received $400 less than the third eldest, and so on. How much did each child receive?

27. A man about to take a new job is offered a choice of two propositions. In the first case he will be paid $2000 per year and get a $200 increase every

year. In the second case he will be paid \$1000 every six months and get a \$50 increase every six months. Which proposition should he accept if he plans to stay 8 years?

28. Write a sequence of three terms that is an A.P. and, at the same time, a G.P.

29. Find an A.P. whose first term is 1, and whose first, second, and sixth terms form a G.P.

30. Find two numbers whose arithmetic mean is 5 and whose harmonic mean is $\frac{16}{5}$.

chapter 12 *Mathematical induction*

98. Induction. Induction is the process of reasoning by which one observes a number of special cases and then draws a general conclusion. For example, children, at an early age, notice that whenever an object is thrown into the air, it always falls to the earth. Laymen generalize these observations by saying, "what goes up must come down." Scientists consider this phenomenon as a result of the law of gravity, which has been proved by induction, i.e., by many, many instances in which the law has been verified without a single exception.

From a mathematician's viewpoint, induction is not a satisfactory means of proof. The mere fact that the truth of a statement has been verified several times does not constitute a mathematical proof of its validity in general. Instead of saying that the law of gravity has been proved, it may be more truthful to say that it never has been disproved.

Let us investigate the following formula

$$2^2 + 4^2 + 6^2 + \cdots + (2n)^2 = n^3 + 4n^2 - 3n + 2.$$

We find,

for $n = 1$: $$2^2 = 1 + 4 - 3 + 2,$$ $$4 = 4; \quad \text{True.}$$

for $n = 2$: $$2^2 + 4^2 = 8 + 16 - 6 + 2,$$ $$20 = 20; \quad \text{True.}$$

for $n = 3$: $$2^2 + 4^2 + 6^2 = 27 + 36 - 9 + 2,$$ $$56 = 56. \quad \text{True.}$$

From these three special cases, one might be tempted to say that the formula is true if n is any positive integer. We can see that induction has led us astray if we set $n = 4$ and obtain the contradiction $120 = 118$. In fact the formula is not true for any values of n except 1, 2, 3.

99. Mathematical induction. Mathematical induction differs from ordinary induction in that it presents "an airtight case," leaving no doubt whatsoever as to the validity of the conclusion. The principle involved is similar to that in the following analogy. Suppose that a number of freight cars are standing on a track. We can prove that all the cars can be moved if we establish these two facts: (I) the first car can be moved, and (II) each car in motion will (by a coupling) compel the following car to move. This type of reasoning is used primarily in proving theorems (formulas or statements) that deal with a positive integral number (usually called n) of cases.

Mathematical induction consists of two parts:

Part I. *A verification of the theorem for one special case, usually* $n = 1$. Although not necessary, other verifications are desirable to familiarize the student with the meaning of the theorem.

Part II. *A proof that if the theorem is true for $n = k$, then it must be true for $n = k + 1$*, i.e., if the theorem is true for any particular value of n, it must be true for the next value of n.

Example 1. Prove by mathematical induction

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

Proof. **Part I.** Verify,

for $n = 1$: $\quad 1^2 = \frac{1}{6}(1)(2)(3) = 1.\quad$ True.
for $n = 2$: $\quad 1^2 + 2^2 = \frac{1}{6}(2)(3)(5) = 5.\quad$ True.
for $n = 3$: $\quad 1^2 + 2^2 + 3^2 = \frac{1}{6}(3)(4)(7) = 14.\quad$ True.

Part II. Let k represent any particular value of n. For $n = k$, the formula becomes

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \tfrac{1}{6}k(k+1)(2k+1). \tag{A}$$

For $n = k + 1$, the formula is

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\ &= \tfrac{1}{6}(k+1)([k+1]+1)(2[k+1]+1) \\ &= \tfrac{1}{6}(k+1)(k+2)(2k+3). \end{aligned} \tag{B}$$

We must show that if the formula is true for $n = k$, then it must be true for $n = k + 1$. In other words, we must show that (B) follows from (A). The left side of (A) can be converted into the left side of (B) by merely adding $(k + 1)^2$. All that remains to be demonstrated is that when $(k + 1)^2$ is added to the right side of (A), the result is the right side of (B):

$$\begin{aligned} &\tfrac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2 \\ &= (k + 1)\left[\frac{k(2k + 1)}{6} + k + 1\right] \\ &= (k + 1)\left[\frac{2k^2 + k + 6k + 6}{6}\right] \\ &= \frac{(k + 1)(2k^2 + 7k + 6)}{6} \\ &= \tfrac{1}{6}(k + 1)(k + 2)(2k + 3). \end{aligned}$$

We have thus established that if (A) is true, then (B) must be true, i.e., if the formula is true for $n = k$, then it must be true for $n = k + 1$.

By verification, we know the formula is true for $n = 3$. Therefore by Part II it must hold for $n = 4$ (using $k = 3$ and $k + 1 = 4$). Since the formula is true for $n = 4$, it must be true for $n = 5$, and so on for all positive integral values of n.

The two parts of the proof are equally important. The "formula" in Art. 98 illustrates that Part I without Part II does not constitute a proof. In terms of the freight-train analogy, we have moved the first three cars but failed to make certain that each car is coupled to the one behind.

In the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1) + 100,$$

we can establish Part II, but Part I is impossible because the "formula" is not true for any value of n. In this case each freight car is coupled to the next one but the first car cannot be moved.

Comment. The term following the kth term in a sequence may be found by replacing k with $(k + 1)$ in the kth term. Thus, if the kth term is $(2k - 1)$, the next term is $(2[k + 1] - 1)$ or $(2k + 1)$.

Exercise 55

Prove the following theorems by mathematical induction.

1. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

2. $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.

3. $2 + 4 + 6 + \cdots + 2n = n(n + 1)$.

4. $3 + 7 + 11 + \cdots + (4n - 1) = n(2n + 1)$.

5. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n + 1)} = \dfrac{n}{n + 1}$.

6. $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \cdots + \dfrac{1}{(2n - 1)(2n + 1)} = \dfrac{n}{2n + 1}$.

7. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$.

8. $1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + \cdots + (2n - 1)(2n) = \frac{1}{3}n(n + 1)(4n - 1)$.

9. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n + 1)^2$.

10. $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(4n^2 - 1)$.

11. $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots + \dfrac{1}{2^n} = \dfrac{2^n - 1}{2^n}$.

12. $1 + 3 + 6 + \cdots + \dfrac{n}{2}(n + 1) = \dfrac{1}{6}n(n + 1)(n + 2)$.

13. $2 + 2^2 + 2^3 + \cdots + 2^n = 2(2^n - 1)$.

14. $a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a - ar^n}{1 - r}$.

15. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n + 2) = \frac{1}{6}n(n + 1)(2n + 7)$.

16. Prove that the arithmetic progression formula $l = a + (n - 1)d$ is true for all positive integral values of n.

17. If n is any positive integer, prove that $x^n - y^n$ is divisible by $x - y$.

Hint. In Part II, use the identity $x^{k+1} - y^{k+1} = x(x^k - y^k) + y^k(x - y)$. Notice that if $x^k - y^k$ is divisible by $x - y$, then the right side of this equation is likewise, and so is the left side, $x^{k+1} - y^{k+1}$.

18. If n is any positive integer, prove that $x^{2n} - y^{2n}$ is divisible by $x + y$.

Hint. In Part II, use the identity $x^{2k+2} - y^{2k+2} = x^2(x^{2k} - y^{2k}) + y^{2k}(x^2 - y^2)$.

chapter 13

The binomial theorem

100. The binomial formula. The following expansions of various powers of the binomial $(a + b)$ may be obtained by actual multiplication.

$$
\begin{aligned}
(a + b)^1 &= a + b.\\
(a + b)^2 &= a^2 + 2ab + b^2.\\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3.\\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.
\end{aligned}
$$

For $n = 1, 2, 3, 4$, we observe that the expansion of $(a + b)^n$ has the following properties:

I. The first term is a^n; in each succeeding term the exponent of a decreases by 1.

II. The second term is $na^{n-1}b$; in each succeeding term the exponent of b increases by 1.

III. If the coefficient of any term is multiplied by the exponent of a and divided by the number of the term, the result is the coefficient of the next term.

Illustration. In the expansion of $(a + b)^4$, the coefficient of the second term is 4. Using property III, we get $\frac{4(3)}{2} = 6$ for the coefficient of the third term.

Assuming these properties are true for all positive integral values of n, we have

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n,$$

which is called the **binomial formula.** In Art. 104, we shall prove that it holds when n is any positive integer. For the present we shall use it without proof.

For checking purposes it will be useful to remember these additional properties of the expansion:

IV. The sum of the exponents of a and b in each term is n.

V. The number of terms is $(n+1)$.

VI. The coefficients are symmetric, i.e., the coefficients of the first and last terms are equal; the coefficients of the second term and the next to the last term are equal, etc.

Example 1. Use the binomial formula to expand $(x^2+2y)^5$.

Solution.

$$\begin{aligned}(x^2+2y)^5 &= (x^2)^5 + 5(x^2)^4(2y) + 10(x^2)^3(2y)^2 + 10(x^2)^2(2y)^3 \\ &\quad + 5(x^2)(2y)^4 + (2y)^5 \\ &= x^{10} + 10x^8y + 40x^6y^2 + 80x^4y^3 + 80x^2y^4 + 32y^5.\end{aligned}$$

Example 2. Expand to four terms and simplify $(r^3-10t)^{20}$.

Solution.

$$\begin{aligned}(r^3-10t)^{20} &= (r^3+[-10t])^{20} \\ &= (r^3)^{20} + 20(r^3)^{19}(-10t) + \frac{\overset{10}{\cancel{20}}\cdot 19}{1\cdot\cancel{2}}(r^3)^{18}(-10t)^2 \\ &\quad + \frac{20\cdot 19\cdot\overset{3}{\cancel{18}}}{1\cdot\cancel{2}\cdot\cancel{3}}(r^3)^{17}(-10t)^3 + \cdots \\ &= r^{60} - 200r^{57}t + 19{,}000r^{54}t^2 - 1{,}140{,}000r^{51}t^3 + \cdots.\end{aligned}$$

Notice that the binomial (r^3-10t) should be written as a *sum* before the binomial formula can be used.

101. Pascal's triangle. The coefficients in the binomial expansion can be arranged in a clever pattern called **Pascal's triangle.** Each

row of the triangle consists of the coefficients in the expansion of the binomial to the left of the row.

$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5 1
$(a+b)^6$	1 6 15 20 15 6 1
	. . . .

Every row begins and ends with 1. Each of the other numbers may be obtained by adding the two numbers that appear immediately above it, one at the left and the other at the right. Thus, in the seventh row $6 = 1 + 5$, $15 = 5 + 10$, $20 = 10 + 10$, etc.

102. Factorial notation. If n is any positive integer, the symbol $n!$ (read ***n* factorial**) is defined to be the product of the integers from 1 to n:

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

Illustration. $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$

$$\frac{9!}{7!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} = 8 \cdot 9 = 72.$$

For convenience, we define 0! to be 1.

Exercise 56

Use the binomial formula to expand; then simplify.

1. $(a+b)^6$. **2.** $(a+b)^8$. **3.** $(x-y)^7$.

4. $(r-s)^5$. **5.** $(r^2-t^3)^4$. **6.** $\left(x^2 - \dfrac{.3}{x}\right)^4$.

7. $(2s^3+1)^5$. **8.** $(10x+y^2)^3$. **9.** $(4x^7+5y)^3$.

10. $\left(\dfrac{2x}{y} + \dfrac{y}{2}\right)^5$. **11.** $\left(\dfrac{x}{y} - \dfrac{y}{2}\right)^4$. **12.** $(3x-2)^4$.

13. $(x^4-10y)^5$. **14.** $(x^{\frac{1}{2}} - y^{\frac{1}{4}})^6$. **15.** $(1+\sqrt{5})^6$.

16. $(\frac{1}{2}+\frac{1}{2})^6$. **17.** $\left(x + \dfrac{1}{x}\right)^8$. **18.** $(r+s+t)^3$.

Hint. Consider $(r+s)$ as a single term: $(r+s+t)^3 = ([r+s]+t)^3$.

Write the first four terms of each of the following expansions.

19. $(a + b)^{40}$. **20.** $(x - y)^{100}$. **21.** $(x^5 - 2y)^{21}$.

22. $(x - \frac{1}{3})^8$. **23.** $(r^6 - 3\sqrt{t})^{10}$. **24.** $(r^{\frac{1}{2}} + s^{\frac{1}{2}})^{14}$.

25. $\left(2 + \dfrac{s}{2}\right)^6$. **26.** $\left(\dfrac{x^2}{y^2} + \dfrac{y}{x}\right)^{13}$. **27.** $(x^4 + 10y)^{12}$.

28. $(x^2 - 3y)^{22}$. **29.** $(- c^2 + d^3)^{11}$. **30.** $(- r^{-2} - b^{-1})^7$.

31. Compute the value of $(1.001)^{21}$ to four decimal places by expanding $(1 + .001)^{21}$ to three terms and evaluating them.

32. Compute the value of $(1.02)^{10}$ to four decimal places by expanding $(1 + .02)^{10}$ to four terms and evaluating them.

33. Compute the value of $(.99)^{12}$ to four decimal places by expanding $(1 - .01)^{12}$ to four terms and simplifying.

34. Compute the value of $(.998)^{20}$ to four decimal places by expanding $(1 - .002)^{20}$ to as many terms as necessary.

Simplify and compute the value of the following.

35. $\dfrac{10!}{12!}$. **36.** $\dfrac{7!}{7}$. **37.** $\dfrac{15!}{12!\,3!}$. **38.** $\dfrac{98!}{97!}$.

Simplify.

39. $\dfrac{(k + 1)!}{k!}$. **40.** $\dfrac{(k - 1)!}{(k + 1)!}$.

103. The general term of the binomial formula. An examination of the binomial formula for $(a + b)^n$ shows that *the term involving b^r is*

$$\frac{n(n - 1)(n - 2) \cdots \text{to } r \text{ factors}}{1 \cdot 2 \cdot 3 \cdots \text{to } r \text{ factors}} a^{n-r}b^r, \tag{1}$$

which is the same as

$$\frac{n(n - 1)(n - 2) \cdots (n - r + 1)}{r!} a^{n-r}b^r. \tag{2}$$

Since the term involving b^r is the $(r + 1)$th term, *the* rth *term is*

$$\frac{n(n - 1)(n - 2) \cdots \text{to } (r - 1) \text{ factors}}{1 \cdot 2 \cdot 3 \cdots \text{to } (r - 1) \text{ factors}} a^{n-r+1}b^{r-1}. \tag{3}$$

Either (1) or (3) may be considered as the general term of the expansion. Of the three formulas, the easiest to remember is (1).

Example 1. Write the 7th term of $(3x + y^{11})^{10}$.

Solution. The 7th term involves the 6th power of y^{11}. Using (1), we get

$$\frac{10 \cdot \overset{3}{\cancel{9}} \cdot \overset{4}{\cancel{8}} \cdot 7 \cdot \cancel{6} \cdot \cancel{5}}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5} \cdot \cancel{6}}(3x)^4(y^{11})^6 = 210(81x^4)y^{66} = 17{,}010x^4y^{66}.$$

104. The binomial theorem. *The binomial formula for $(a + b)^n$ is true if n is any positive integer.*

Proof. We shall use mathematical induction.

Part I. The formula has been verified for $n = 1, 2, 3, 4$.

Part II. We shall show that if the formula is true for $n = k$, then it must be true for $n = k + 1$. For $n = k$,

$$(a + b)^k = a^k + ka^{k-1}b + \cdots + \frac{k(k-1)\cdots(k-r+2)}{1 \cdot 2 \cdots (r-1)}a^{k-r+1}b^{r-1} + \frac{k(k-1)\cdots(k-r+1)}{1 \cdot 2 \cdots r}a^{k-r}b^r + \cdots + b^k. \quad (1)$$

Multiply both sides of this equation by $(a + b)$. On the right side of the resulting equation we shall exhibit only the first two terms, the last term, and the term involving b^r.

$$\begin{aligned}
&(a+b)^{k+1} = \\
{*}\quad & a^{k+1} + ka^kb + \cdots + \frac{k(k-1)\cdots(k-r+1)}{1 \cdot 2 \cdots r}a^{k-r+1}b^r + \cdots \\
\dagger\quad & \qquad + a^kb + \cdots + \frac{k(k-1)\cdots(k-r+2)}{1 \cdot 2 \cdots (r-1)}a^{k-r+1}b^r + \cdots + b^{k+1} \\
\hline
& a^{k+1} + (k+1)a^kb + \cdots + \frac{(k+1)k\cdots(k-r+2)}{1 \cdot 2 \cdots r}a^{k-r+1}b^r + \cdots + b^{k+1}.
\end{aligned}$$

We see that this result is exactly the same as the expansion of $(a + b)^n$ if n is replaced by $(k + 1)$. (Compare the general term with (2) in Art. 103.) This shows that if the formula is true for $n = k$, then it must be true for $n = k + 1$, regardless of what k is. We have verified the formula for $n = 4$; hence it must be true for $n = 5$. Being true for $n = 5$, it must be true for $n = 6$; and so on for all positive integral values of n.

* This line results when the right side of (1) is multiplied by a.

† This line is the product of the right side of (1) by b.

Comment. The simplification of the coefficient of the term involving b^r is performed as follows:

$$\frac{k(k-1)\cdots(k-r+1)}{1\cdot 2\cdots r}+\frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}$$
$$=\frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}\left[\frac{k-r+1}{r}+1\right]$$
$$=\frac{k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)}\left[\frac{k+1}{r}\right]$$
$$=\frac{(k+1)k(k-1)\cdots(k-r+2)}{1\cdot 2\cdots(r-1)\cdot r}$$
$$=\frac{(k+1)k\cdots(k-r+2)}{1\cdot 2\cdots r}.$$

105. The binomial formula for negative and fractional exponents. If n is not a positive integer, the binomial formula for $(a+b)^n$ gives an expansion with an infinite number of terms. It can be shown that the formula holds for all real values of n provided $|b|<|a|$. This means that as we include more and more terms of the expansion their sum will more closely approach the value on the left side. (Compare with Art. 95.)

Example 1. Expand to three terms and simplify: $(8+y)^{\frac{2}{3}}$.

Solution.

$$(8+y)^{\frac{2}{3}}=8^{\frac{2}{3}}+\tfrac{2}{3}\cdot 8^{-\frac{1}{3}}\cdot y+\frac{\frac{2}{3}(-\frac{1}{3})}{1\cdot 2}\cdot 8^{-\frac{4}{3}}\cdot y^2+\cdots$$
$$=4+\frac{y}{3}-\frac{y^2}{144}+\cdots.$$

Example 2. Approximate $\sqrt[5]{.95}$ by using three terms of the binomial formula.

Solution.

$$\sqrt[5]{.95}=(1+[-.05])^{\frac{1}{5}}$$
$$=1^{\frac{1}{5}}+\frac{1}{5}\cdot 1^{-\frac{4}{5}}(-.05)+\frac{\frac{1}{5}(-\frac{4}{5})}{1\cdot 2}1^{-\frac{9}{5}}(-.05)^2+\cdots$$
$$=1+\tfrac{1}{5}(-.05)-\tfrac{2}{25}(.0025)+\cdots$$
$$=1-.01-.0002+\cdots$$
$$=.9898.$$

Exercise 57

1. Find the 4th term of $(a + w^5)^{11}$.
2. Find the 5th term of $(a - \sqrt{2})^{13}$.
3. Find the 6th term of $\left(\frac{x}{2} - \frac{2}{y}\right)^9$.
4. Find the 8th term of $(4r - t^2)^{10}$.
5. Find the term involving y^7 in $(x - y)^{15}$.
6. Find the term involving b^5 in $(a + b)^{12}$.
7. Find the term involving y^{30} in $(x + y^{10})^8$.
8. Find the term involving x in $(2x - y^5)^{14}$.
9. Find the middle term of $(2x + 3y)^6$.
10. Find the middle terms of $(x^2 - 10)^7$.

Expand to four terms and simplify.

11. $(1 + x)^{-1}$. 12. $(1 + x)^{-2}$. 13. $(1 - x)^{-2}$. 14. $(1 + x)^{-\frac{1}{2}}$.

15. $(x^2 + y)^{\frac{1}{2}}$. 16. $(r^6 - t)^{\frac{1}{3}}$. 17. $(25 + y)^{\frac{3}{2}}$. 18. $(100 - x^2)^{\frac{1}{2}}$.

Approximate the following radicals by using the first three terms of the binomial formula.

19. $\sqrt[3]{1.03}$. 20. $\sqrt[4]{1.08}$. 21. $\sqrt[5]{1.05}$.

22. $\sqrt[3]{1.06}$. 23. $\sqrt[4]{.96}$. 24. $\sqrt[5]{.99}$.

Approximate the following radicals by using the first two terms of the binomial formula.

25. $\sqrt[3]{8.6}$. 26. $\sqrt[6]{64.96}$. 27. $\sqrt[5]{32.8}$. 28. $(81.24)^{\frac{3}{4}}$.

chapter 14 *Inequalities*

106. Inequalities. An **inequality** is a statement that one real quantity is greater than or less than another real quantity. The statement $a > b$ (read "a is greater than b") means that $(a - b)$ is a positive number. If a and b are plotted on the directed line in Fig. 21, then

FIG. 21

$a > b$ means that a is to the *right* of b. Similarly, the statement $c < d$ (read "c is less than d") means that $(c - d)$ is a negative number. In Fig. 21, c would lie to the *left* of d. The statement $2 < x \leqq 5$ (read "x is greater than 2 and less than or equal to 5") means that x is any number between 2 and 5, including 5 but not 2. The inequality $|x| < 1$ (read "the absolute value of x is less than 1") means that x is between -1 and 1: $-1 < x < 1$.

An **absolute inequality** is one that is satisfied by all real values of the letters involved. Thus, $x^2 \geqq 0$ is an absolute inequality because the square of any real number is either positive or 0. A **conditional inequality** is one that is satisfied only by certain values of the letters involved. Thus, $x - 5 > 0$ is satisfied only by numbers greater than 5.

107. Properties of inequalities. Two inequalities, such as $a < b$ and $c < d$, in which the inequality signs point in the same direction are said to have the same sense.

In dealing with inequalities, we must observe the following principles.

I. The sense of an inequality is not changed if the same number is added to or subtracted from both sides.

Illustration 1.

$$11 > 7.$$
$$11 + 5 > 7 + 5.$$

Illustration 2.

If $x + 2 \leqq 8$,
then $x \leqq 6$.

II. The sense of an inequality is not changed if both sides are multiplied or divided by the same **positive** *number.*

Illustration 3.

$$5 < 6.$$
$$15 < 18.$$

Illustration 4.

If $4x > 12$,
then $x > 3$.

III. The sense of an inequality is **reversed** *if both sides are multiplied or divided by the same* **negative** *number.*

Illustration 5. $7 > 6.$

Multiply by -1 and reverse sense: $-7 < -6.$

Illustration 6. If $-3x < 6$,

Divide by -3 and reverse sense: then $x > -2$.

IV. The sense of an inequality of positive numbers is not changed if the same positive power or the same positive root of each side is taken.

Illustration 7. $4 > 3.$

Square both sides: $16 > 9.$

Illustration 8. $25 < 36.$

Take square roots: $5 < 6.$

The proofs of these properties are similar and follow almost immediately from the definition of an inequality.

Proof of I. We shall prove that the inequality $a > b$ is not changed in sense if c is added to both sides. Since $a > b$, their difference $(a - b)$ is equal to some positive number p:

$$a - b = p.$$

Add and subtract c on left side: $a + c - b - c = p.$

Group terms: $(a + c) - (b + c) = p.$

Hence $a + c > b + c,$

by the definition of an inequality.

In a similar manner we can show that $a - c > b - c$.

The proofs of the other three properties follow the same general pattern.

108. Solution of conditional inequalities. We shall confine our discussion to inequalities that involve only one unknown. To solve an inequality means to find the values or ranges of values for which the inequality is satisfied.

Linear inequalities may be solved algebraically by isolating the unknown on one side. This is done by applying the properties of inequalities.

Example 1. Solve: $\frac{x}{2} - 6 < 3x + 4.$

Solution.

Multiply by 2: $x - 12 < 6x + 8.$

Add 12 and subtract $6x$: $x - 6x < 12 + 8.$

$$-5x < 20.$$

Divide by -5 and reverse the sense: $x > -4.$

The given inequality is satisfied by all values of x greater than -4. The student should test it for a few special values such as -3, 0, 100.

Quadratic inequalities (and inequalities of higher degree) may be solved graphically.

Example 2. Solve: $x^2 < 2x + 2.$

Solution. Make the right side zero by using property I.

$$x^2 - 2x - 2 < 0.$$

Graph the left side as a function of x.

x		-1	0	1	2	3	
$f(x) = x^2 - 2x - 2$		1	-2	-3	-2	1	

Obviously $f(x) < 0$ when the curve (Fig. 22) lies *below* the x-axis. i.e., for values of x between $-.7$ and 2.7.

Hence, $-.7 < x < 2.7.$

Since these results were read from the curve, they are only approximations. If the student should read 2.6 or 2.8 instead of 2.7, his result would be acceptable.

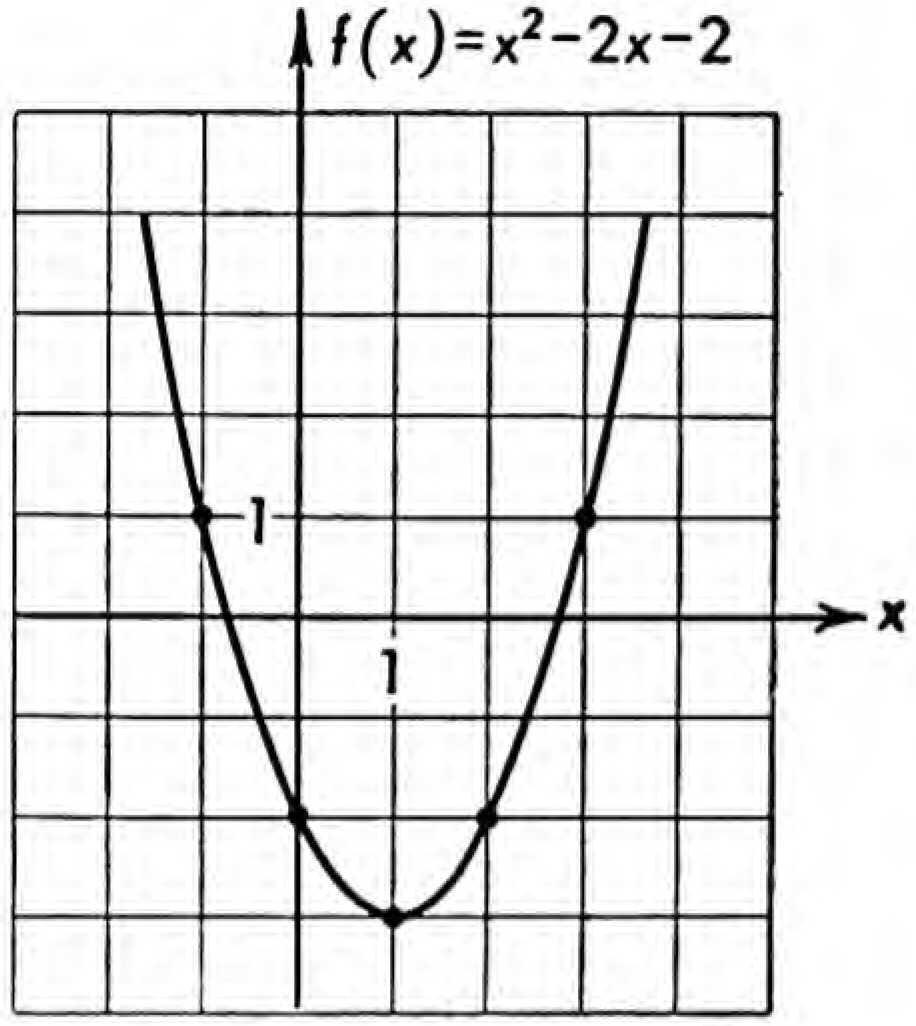


FIG. 22

In case more accuracy is desired, we can solve the corresponding *equation* $x^2 - 2x - 2 = 0$ and find $x = 1 \pm \sqrt{3}$. Thus the curve crosses the x-axis when $x = 1 - \sqrt{3}$ and $x = 1 + \sqrt{3}$, and the inequality is true for

$$1 - \sqrt{3} < x < 1 + \sqrt{3}.$$

Example 3. Solve

$$x^2 - 2x - 2 > 0.$$

Solution. The graph of $(x^2 - 2x - 2)$ is shown in Fig. 22. The inequality holds when the curve is *above* the x-axis, i.e., when

$$x > 2.7 \quad \text{or} \quad x < -.7.$$

Example 4. Rewrite the following inequality without using an absolute value sign: $\quad |x - 5| < 1.$

Solution. Since the absolute value of $(x - 5)$ is less than 1, we know that $(x - 5)$ lies between -1 and 1:

$$-1 < x - 5 < 1.$$

Add 5: $\quad 4 < x < 6.$

Example 5. Solve: $\quad x^2 > 16.$

Solution. We know that if a number is in absolute value less than 4, then its square is less than 16. Likewise, the square of any number whose absolute value is greater than 4 must be greater than 16. Hence

if $\quad x^2 > 16,$

then $\quad |x| > 4,$

and $\quad x > 4 \quad \text{or} \quad x < -4.$

109. Proofs of absolute inequalities. To prove an absolute inequality, we must (1) start with an inequality we know is true,

(2) apply the properties of inequalities, and (3) derive the inequality we are to prove. An analysis of the inequality to be proved will usually give us a clue as to the inequality with which we should start.

Example 1. If $x > 0$, $y > 0$, $x \neq y$, prove that

$$\frac{x+y}{2} > \sqrt{xy}.$$

Analysis. If $\dfrac{x+y}{2} > \sqrt{xy}$, then $x + y > 2\sqrt{xy}$, and $(x+y)^2 > (2\sqrt{xy})^2$, and $x^2 + 2xy + y^2 > 4xy$, and $x^2 - 2xy + y^2 > 0$, and $(x-y)^2 > 0$.

Proof. Since $x \neq y$, we know that

$$(x-y)^2 > 0.$$

Expand: $$x^2 - 2xy + y^2 > 0.$$

Add $4xy$: $$x^2 + 2xy + y^2 > 4xy.$$

Take positive square roots: * $$x + y > 2\sqrt{xy}.$$

Divide by 2: $$\frac{x+y}{2} > \sqrt{xy}.$$

We have shown that the arithmetic mean of x and y is larger than their geometric mean.

Exercise 58

Solve the following linear inequalities.

1. $4x - 3 < 5$.

2. $2x - 9 > x$.

3. $\frac{1}{5}x > 10$.

4. $3 - 2x < 7$.

5. $5x + 3 > 7x - 9$.

6. $2x - 4 > 5x + 8$.

7. $2 - 5x < 4x - 1$.

8. $4 - 5x < 1 - 3x$.

9. $1 - \frac{1}{2}x < \frac{1}{4}x + \frac{5}{6}$.

10. $\frac{1}{3}x + 5 < \frac{1}{2}x + 7$.

Solve graphically.

11. $x^2 > 9$.

12. $x^2 < 4$.

13. $x^2 < 2x$.

14. $x^2 > 2x + 8$.

* This is permissible because both sides of the preceding inequality are positive since $x > 0$ and $y > 0$.

15. $x^2 < 5x - 4.$

16. $x^2 + x > 6.$

17. $x^2 > 4x + 2.$

18. $x^2 - 6x + 7 < 0.$

19. $1 + x - 2x^2 < 0.$

20. $2x^2 - 4x + 1 < 0.$

21. $x^3 - 3x^2 - x + 3 > 0.$

22. $x^3 - 3x^2 + 1 < 0.$

23. $x^3 + x - 3 < 0.$

24. $(x - 1)(x - 3)(x - 5) > 0.$

Rewrite without using absolute value signs and solve for x.

25. $|x - 3| < 4.$

26. $|x - 2| > 6.$

27. $|x + 1| \geqq 7.$

28. $|4 - x| \leqq 1.$

Solve by inspection without graphing.

29. $x^2 > 49.$

30. $x^2 < 81.$

31. $4x^2 \leqq 9.$

32. $9x^2 > 25.$

33. Prove that if $x \neq y$, then $x^2 + y^2 > 2xy$.

34. Prove that if $x > 0$ and $x \neq 1$, then $x + \dfrac{1}{x} > 2$.

35. Prove that if $x + y > 0$ and $x \neq y$, then $\dfrac{x + y}{2} > \dfrac{2xy}{x + y}$.

36. Prove that if $x > y > 0$, then $x^3 - y^3 > (x - y)^3$.

37. Find the values of k for which the following quadratic equation in x will have roots that are real and unequal: $x^2 - 2kx + 2k = 0$.

Hint. The discriminant must be positive.

38. Find the values of k for which the following quadratic equation in x will have imaginary roots: $kx^2 - 5x + 3 = 0$.

Hint. The discriminant must be negative.

39. Find the values of k for which the straight line $y = x + k$ will meet the circle $x^2 + y^2 = 18$ in two distinct points.

Hint. Eliminate y by substituting from the linear into the quadratic equation. The resulting quadratic in x must have real, unequal roots, i.e., its discriminant must be positive.

40. If $a > b$, under what conditions can we say that $\dfrac{1}{a} < \dfrac{1}{b}$?

chapter 15 Complex numbers

110. Complex numbers. In Art. 59 we formulated the following definitions.

1. *Definition.* $i = \sqrt{-1}.$
Consequence. $i^2 = -1.$

2. *A* **pure imaginary number** *is a square root of a negative number.*

3. *A* **complex number** *is a number of the form* $\boldsymbol{a + bi}$, *where* a *and* b *are real numbers and* $i = \sqrt{-1}$. Since zero is a real number, we see that if $a = 0$, the complex number $a + bi$ becomes bi, a pure imaginary number. If $b = 0$, the complex number $a + bi$ becomes a, a real number. Therefore complex numbers include real numbers and pure imaginary numbers as special cases.

4. *An imaginary number is a complex number of the form* $a + bi$, *where* $b \neq 0$.

Illustrations.

Real numbers.	$4,\ \frac{1}{7},\ 0,\ -\frac{2}{9},\ \sqrt[3]{5},\ -\sqrt{2},\ \pi.$
Pure imaginary numbers.	$3i,\ \sqrt{-2},\ -4i,\ -\sqrt{-7}.$
Imaginary numbers.	$2 + 5i,\ -7 + \sqrt{-3},\ \sqrt{2} - i\sqrt{5},\ 3i.$

All these numbers are complex numbers.

The following additional definitions will be useful in later discussions.

5. *The complex numbers* $a + bi$ *and* $a - bi$ *are said to be* **conjugates** *of each other*. Notice that the roots of the equation $x^2 - 6x +$

$25 = 0$ are the conjugate imaginary numbers $3 + 4i$ and $3 - 4i$. It will be shown (Art. 126) that if an imaginary number $a + bi$ is a root of an equation with real coefficients, then the conjugate imaginary $a + bi$ is also a root of this equation.

6. *Two complex numbers are said to be equal provided their real parts are equal and their imaginary parts are equal.* This means that

if $$a + bi = c + di,$$
then $$a = c \quad \text{and} \quad b = d.$$

As a consequence of this definition, we see that

if $$a + bi = 0,$$
then $$a = 0 \quad \text{and} \quad b = 0.$$

111. Algebraic operations on complex numbers. Before performing any algebraic operations on complex numbers, we should write them in the form $a + bi$. (See Art. 59.) Treat i like any other number, but replace i^2 (whenever it appears) with -1.

1. *Addition and subtraction.*

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i.$$

Illustration 1.

$$(2 + 3i) + (7 - 4i) = (2 + 7) + (3 - 4)i = 9 - i.$$
$$(-7 - 5i) - (6 + i) = (-7 - 6) + (-5 - 1)i = -13 - 6i.$$

2. *Multiplication.*

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i.\end{aligned}$$

Illustration 2.

$$\begin{aligned}(3 + 8i)(5 + 7i) &= 15 + 21i + 40i + 56i^2 \\ &= -41 + 61i.\end{aligned}$$

3. *Division.* The quotient of two complex numbers may be expressed in the form $a + bi$ by multiplying the numerator and denominator by *the conjugate of the denominator;*

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2}.$$

Illustration 3.

$$\frac{5+7i}{1-3i}=\frac{(5+7i)(1+3i)}{(1-3i)(1+3i)}=\frac{5+15i+7i+21i^2}{1-9i^2}$$
$$=\frac{-16+22i}{10}=-\frac{8}{5}+\frac{11}{5}i.$$

The division may be checked by multiplying $(-\frac{8}{5}+\frac{11}{5}i)$ by $(1-3i)$. What should the result be?

Exercise 59

Perform each of the indicated operations and express the result in the form $a+bi$.

1. $(7-9i)+(4+5i)$.
2. $(8+i)+(-2+3i)$.
3. $(4-\sqrt{-25})+(\sqrt[3]{5}-\sqrt{-36})$.
4. $(5+\sqrt{-18})+(6+\sqrt{-8})$.
5. $\sqrt{-9}+5\sqrt{-16}-\sqrt{-49}$.
6. $(6-5i)-(4-7i)$.
7. $i-(4+3i)-(-6-i)$.
8. $(1+i)+(3-8i)+(-2+9i)$.
9. $(3-5i)-(4-i)+(6+7i)$.
10. $\sqrt{-12}\,\sqrt{-3}$.
11. $(2\sqrt{-3})^2$.
12. $(2+i)(3+5i)$.
13. $(6-7i)(8-i)$.
14. $(4-3i)(-2-i)$.
15. $(2\sqrt{3}+\sqrt{-5})(\sqrt{3}+\sqrt{-20})$.
16. $(4-\sqrt{-7})(4+\sqrt{-7})$.
17. $(5-6i)^2$.
18. $(3-\sqrt{-5})^2$.
19. $(2-3\sqrt{-7})^2$.
20. $(2+5i)^3$.
21. $(1+i)^4$.
22. $(1+2i)(3+4i)(5+6i)$.
23. $i^{10}+i^{33}$.
24. $i^{47}+i^{48}+i^{49}$.
25. $\dfrac{14-5i}{2+3i}$.
26. $\dfrac{26+15i}{4+i}$.
27. $\dfrac{2-\sqrt{-9}}{7+i}$.
28. $\dfrac{4-8i}{1-2i}$.
29. $\dfrac{\sqrt{-8}}{3+\sqrt{-2}}$.
30. $\dfrac{7+8i}{i}$.
31. $\dfrac{5i}{3-4i}$.
32. $\dfrac{2}{(1+i)^2}$.
33. $\dfrac{(6-7i)^2}{i(3+2i)}$.
34. $\dfrac{(3+5i)(1+2i)}{(4+i)(3-i)}$.

Express the reciprocal of the number in the form $a + bi$.

35. $4 + i$.
36. $5 - 6i$.
37. $-i$.
38. $\sqrt{-20} + \sqrt{-45}$.

State the conjugate of each complex number.

39. $-5 - 7i$.
40. 5.
41. $11i$.
42. $6 + 2\sqrt{-5}$.

Find the values of the real numbers x and y.

43. $2x + yi = 6 + 7i$.
44. $5x - 3yi = 20 + 39i$.
45. $(x + yi)(1 + 3i) = -19 + 13i$.
46. $(x + i)(2 + yi) = 7 + 17i$.
47. $(2x - 8) - (y - 9)i = 0$.
48. Prove that the sum of two conjugate complex numbers is a real number.
49. Prove that the product of two conjugate complex numbers is a positive real number.
50. Show that $\left(\dfrac{a + bi}{c + di} + \dfrac{a - bi}{c - di}\right)$ is a real number.

112. Graphical representation of complex numbers. Let us represent the real numbers by points on a horizontal directed line * (Fig. 23). Let the vector V represent the *directed* segment connecting the origin O to the point corresponding to the real number a.

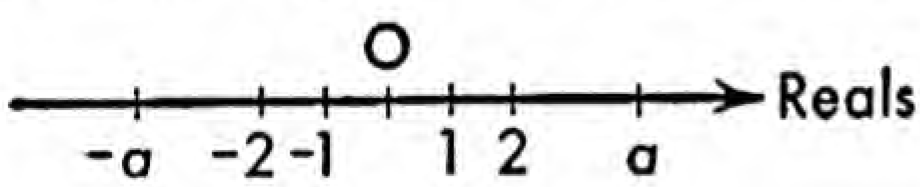


FIG. 23

Since $ai^2 = -a$, it can be said that multiplying a by $i \cdot i$ is geometrically equivalent to rotating V through 180° about O. Consequently it is logical to represent the multiplication of a by i as a rotation of V through 90° about O. Accordingly, the number ai will be represented as a point a units from O on the *vertical* line through O. We shall refer to the horizontal axis as the **axis of reals** and the vertical axis as the **axis of (pure) imaginaries.** This system of axes defines a region called the **complex plane.** It is to be noted that, while the unit on the axis of reals is the number 1, the unit on the axis of imaginaries is the imaginary number i. Hence the complex number $(a + bi)$ is represented by the point a units from the axis of imaginaries and b units from the axis of reals. Figure 24 illustrates

* As in the case of the x-axis of a rectangular coordinate system.

the graphical representation of complex numbers in the complex plane.

It is sometimes convenient to think of the complex number $(a + bi)$ as representing the vector OP (Fig. 24), instead of the point P.

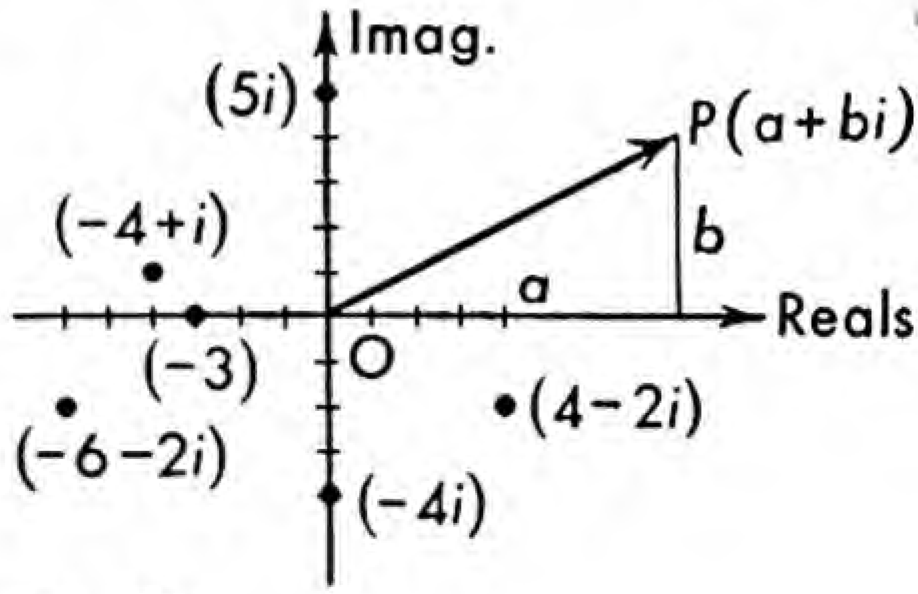


FIG. 24

113. Graphical addition of complex numbers. *To add graphically the complex numbers $a + bi$ and $c + di$:* Let points P and Q represent the two numbers, respectively. Connect the origin O with P and Q. Complete the parallelogram having OP and OQ as adjacent sides; let S be the fourth vertex. Then S represents the sum of the two given complex numbers.

Proof. In Fig. 25, QL, PM, and SN are drawn perpendicular to

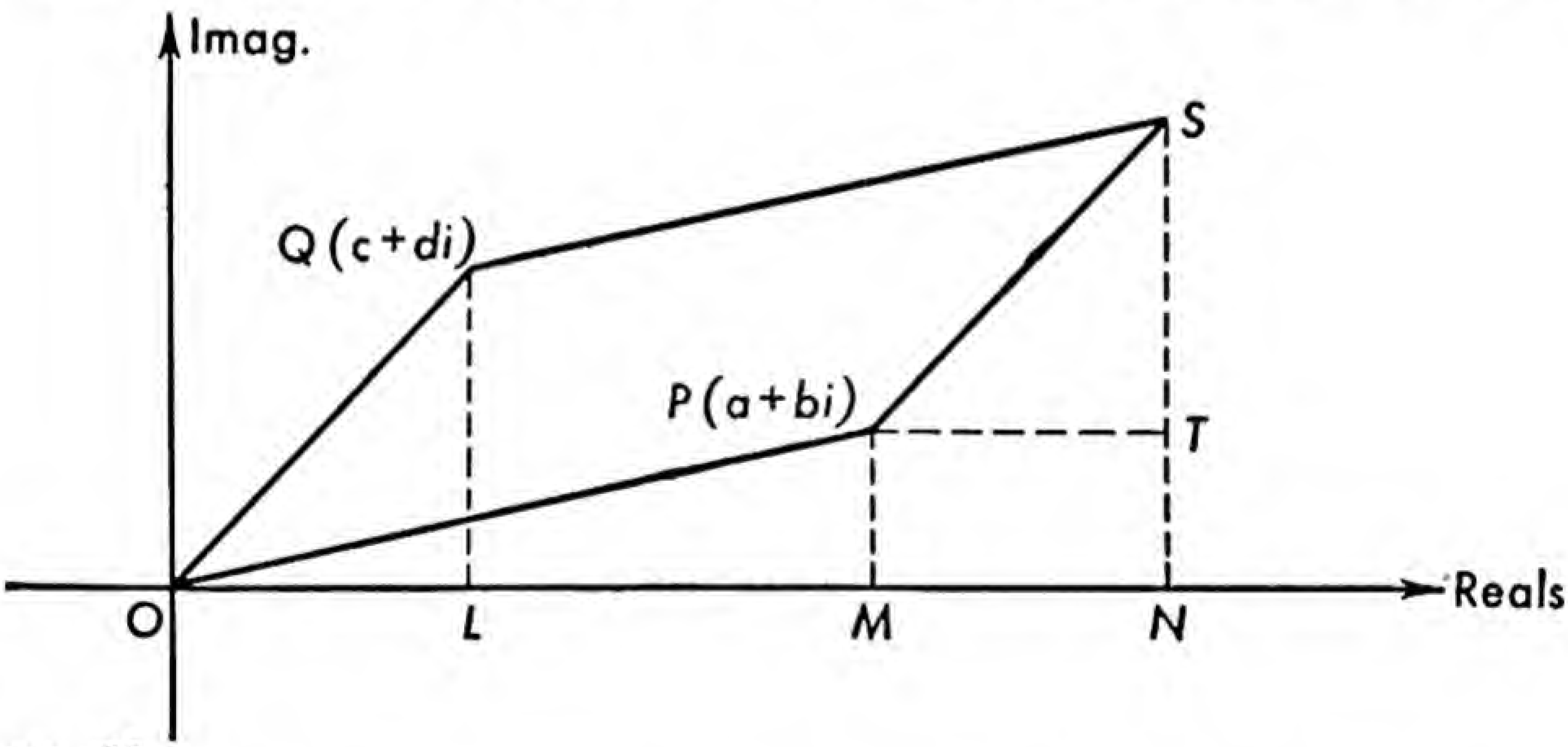


FIG. 25

the axis of reals, and PT is drawn perpendicular to SN. Then triangles OLQ and PTS are congruent. Why? Hence $OL = PT = MN$. And $LQ = TS$. Therefore

$$ON = OM + MN = OM + OL = a + c$$

and $$NS = NT + TS = MP + LQ = b + d.$$

Hence S represents the complex number $(a + c) + (b + d)i$, which is the sum of the complex numbers $a + bi$ and $c + di$.

Three complex numbers can be added graphically by first obtaining the sum of two of them and then adding this to the third.

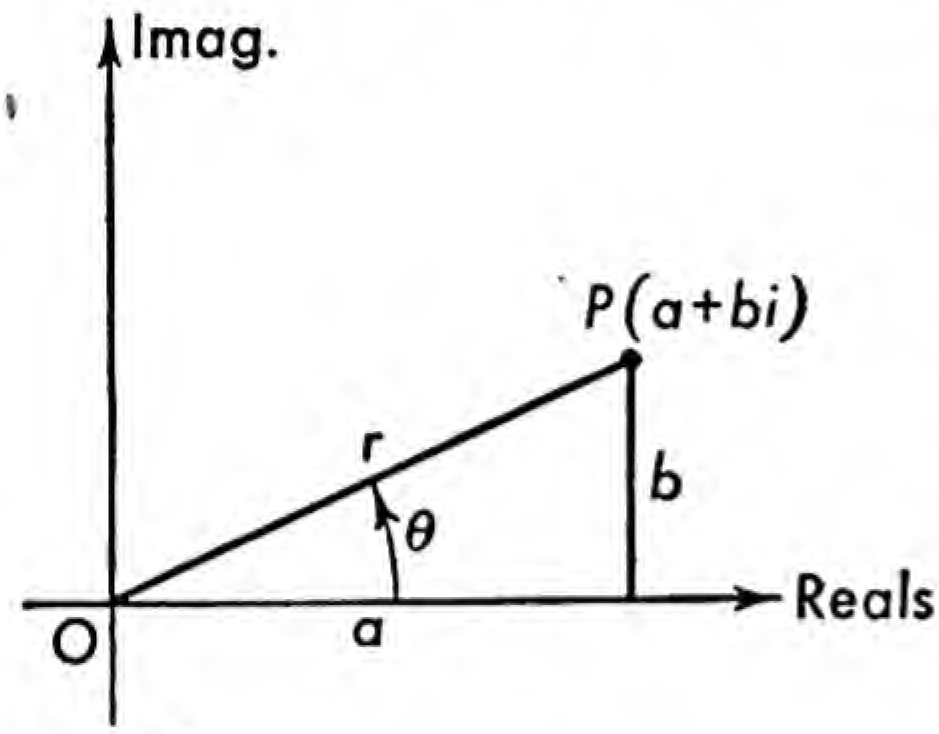


FIG. 26

We can subtract $(c + di)$ from $(a + bi)$ graphically by adding $(a + bi)$ to $(-c - di)$.

114. Trigonometric form of a complex number. Let point P in the complex plane represent the complex number $a + bi$. The **absolute value** * of $a + bi$ is the distance r from O to P. It is always considered positive. The **amplitude** * of $a + bi$ is the angle measured from the positive axis of reals to the line OP. From Fig. 26, it is apparent that

$$(1) \qquad r = \sqrt{a^2 + b^2}, \quad \tan \theta = \frac{b}{a},$$

and

$$(2) \qquad a = r \cos \theta, \qquad b = r \sin \theta.$$

These equations hold regardless of the quadrant in which P lies. If the last equation is multiplied by i and added to the preceding one, we get

$$\boldsymbol{a + bi = r(\cos \theta + i \sin \theta).}$$

The expression $r(\cos \theta + i \sin \theta)$ is called the **trigonometric** † **form** of a complex number. The expression $a + bi$ is called the **algebraic form** of a complex number. The trigonometric form is useful in finding powers and roots of complex numbers.

Any complex number in algebraic form can be expressed in trigonometric form by use of equations (1). After the value of $\tan \theta$ has been obtained, θ can be found by use of a table of trigonometric functions (Table II). In general, there are two angles between 0° and 360° having the same tangent. In order to be certain to get the correct angle, we should *always plot the complex number* ‡ in the com-

* *Absolute value* is also called **modulus;** *amplitude* is sometimes called **argument.**

† Also called the **polar form.** It is sometimes written in the abbreviated form $r \operatorname{cis} \theta$ or $r \underline{|\theta}$.

‡ The expression *plot the complex number* is an abbreviation we shall use for the more rigorous statement, *plot the point corresponding to the complex number.*

plex plane. The amplitude of a real number or a pure imaginary number can be obtained by inspection of its location in the complex plane. For example, the amplitude of $-4i$ is $270°$ (Fig. 24).

Any complex number in trigonometric form can be expressed in algebraic form by replacing $\sin\theta$ and $\cos\theta$ with their numerical values and then simplifying. Thus, $4(\cos 100° + i \sin 100°) = 4(-.174 + .985i) = -.696 + 3.940i$, using Table II.

Example 1. Express each of the following in trigonometric form: (*a*) $4 - 4i$, (*b*) -3.

Solution. (*a*) Plot the number in the complex plane. Equations (1)

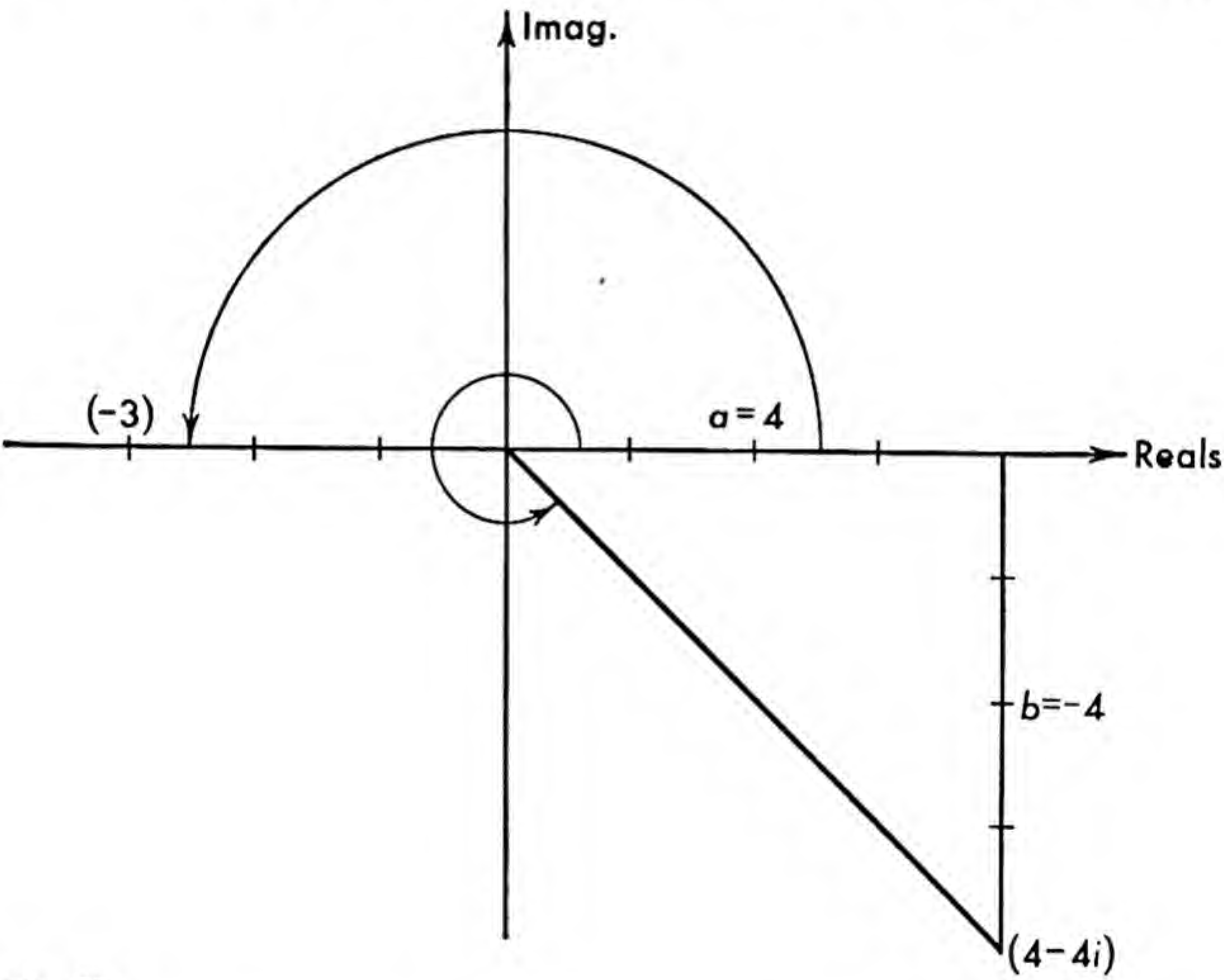


FIG. 27

give us $r = \sqrt{32} = 4\sqrt{2}$, $\tan\theta = -\frac{4}{4} = -1$. From the last equation, θ could be $135°$ or $315°$. From Fig. 27, we see that θ must be $315°$. Hence

$$4 - 4i = 4\sqrt{2}(\cos 315° + i \sin 315°).$$

This result can be checked by replacing $\cos 315°$ and $\sin 315°$ with

$\frac{\sqrt{2}}{2}$ and $-\frac{\sqrt{2}}{2}$, respectively, and then demonstrating that the right side is actually equal to the left side.

(*b*) After plotting the number (Fig. 27), we find by inspection that $r = 3$ and $\theta = 180°$. Hence

$$-3 = 3(\cos 180° + i \sin 180°).$$

It is to be noted that, regardless of the signs of a and b, *r is always positive and the signs in front of $\cos \theta$ and $i \sin \theta$ are always positive.*

Exercise 60

Perform the indicated operations graphically and check the results algebraically.

1. $(4 + i) + (1 + 5i)$.
2. $(7 - 2i) + (1 + 6i)$.
3. $(-2 - 3i) + (4 - 5i)$.
4. $(1 + 3i) + (-5 + 2i)$.
5. $(5 - 6i) - (3 + i)$.
6. $(-6 + 3i) - (-6 - i)$.
7. $(1 + i) + (-2 + 5i) + (-4 - i)$.
8. $(3 - i) + (1 + 2i) - (4 - i)$.

Plot each of the following complex numbers and then express it in trigonometric form.

9. $5 - 5i$.
10. $2 + 2i$.
11. $-\sqrt{2} + i\sqrt{2}$.
12. $-3 - 3i$.
13. -5.
14. $-7i$.
15. $4i$.
16. 3.
17. $1 + i\sqrt{3}$.
18. $2 - 2i\sqrt{3}$.
19. $-3 - i\sqrt{3}$.
20. $-\sqrt{3} + i$.
21. $-5 + 12i$.
22. $-3 - 4i$.
23. $1 - 2i$.
24. $2 + 3i$.

Plot each of the following complex numbers and then express it in algebraic form.

25. $7(\cos 270° + i \sin 270°)$.
26. $5(\cos 180° + i \sin 180°)$.
27. $8(\cos 120° + i \sin 120°)$.
28. $10(\cos 240° + i \sin 240°)$.
29. $\sqrt{3}(\cos 300° + i \sin 300°)$.
30. $4(\cos 30° + i \sin 30°)$.
31. $10(\cos 225° + i \sin 225°)$.
32. $6(\cos 315° + i \sin 315°)$.
33. $5(\cos 63° + i \sin 63°)$.
34. $\cos 100° + i \sin 100°$.

35. What is the amplitude (*a*) of a positive real number? (*b*) of a negative real number? (*c*) of bi if $b > 0$? (*d*) of bi if $b < 0$?

36. Show that the conjugate of $r(\cos\theta + i\sin\theta)$ is $r(\cos[-\theta] + i\sin[-\theta])$.

On one system of coordinates, plot and label the number, its conjugate, and its negative.

37. $3 - 7i$. **38.** $5i$. **39.** 6. **40.** $1 + \sqrt{-16}$.

115. Multiplication and division of complex numbers in trigonometric form.

Theorem 1. The absolute value of the product of two complex numbers is the product of their absolute values; the amplitude of the product is the sum of their amplitudes;

$$\begin{aligned} &\mathbf{r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)} \\ &= \mathbf{r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]}. \end{aligned}$$

Proof. Let $r_1(\cos\theta_1 + i\sin\theta_1)$ and $r_2(\cos\theta_2 + i\sin\theta_2)$ be any two complex numbers in trigonometric form. Their product is

$$\begin{aligned} &r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2[\cos\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2] \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]. \end{aligned}$$

Illustration 1.

$$\begin{aligned} &5(\cos 110^\circ + i\sin 110^\circ)\cdot 7(\cos 160^\circ + i\sin 160^\circ) \\ &= 5\cdot 7[\cos(110^\circ + 160^\circ) + i\sin(110^\circ + 160^\circ)] \\ &= 35[\cos 270^\circ + i\sin 270^\circ] \\ &= 35[0 + i(-1)] = -35i. \end{aligned}$$

This theorem may be extended to include the product of any number of complex numbers:

$$\begin{aligned} &r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2)\cdots r_n(\cos\theta_n + i\sin\theta_n) \\ &= r_1r_2\cdots r_n[\cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)]. \end{aligned}$$

Theorem 2. The absolute value of the quotient of two complex numbers is the quotient of their absolute values; the amplitude of the quotient is the difference of their amplitudes;

$$\mathbf{\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]}.$$

Proof. Multiply top and bottom of the given fraction by $(\cos\theta_2 - i\sin\theta_2)$:

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)\cdot(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)\cdot(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1}{r_2}\cdot\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}$$
$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

Illustration 2.

$$\begin{aligned}\frac{12(\cos 85^\circ + i\sin 85^\circ)}{3(\cos 50^\circ + i\sin 50^\circ)} &= 4(\cos 35^\circ + i\sin 35^\circ)\\ &= 4(.819 + .574i)\ ^*\\ &= 3.276 + 2.296i.\end{aligned}$$

The result in trigonometric form is exact, but the algebraic result is only approximate since the table gives the values of the functions correct to only three decimal places.

116. De Moivre's theorem. *If n is any real number,*

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

This theorem can be proved for positive integral values of n by using the extended form of theorem 1 of Art. 115, with each factor of the left side set equal to $r(\cos\theta + i\sin\theta)$.

It can be shown that De Moivre's theorem is true for all real values of n. We shall use it for only two cases: (1) when n is a positive integer, and (2) when n is the reciprocal of a positive integer. The proof of the latter case is omitted in this book.

Example 1. Use De Moivre's theorem to find the value of $(-\sqrt{3} + i)^7$.

Solution. Plot the number $(-\sqrt{3} + i)$ and then put it in trigonometric form:

$$-\sqrt{3} + i = 2(\cos 150^\circ + i\sin 150^\circ).$$

* From Table II.

Apply De Moivre's theorem:

$$\begin{aligned}(-\sqrt{3}+i)^7 &= [2(\cos 150^\circ + i \sin 150^\circ)]^7 \\ &= 2^7(\cos 7 \cdot 150^\circ + i \sin 7 \cdot 150^\circ) \\ &= 128(\cos 1050^\circ + i \sin 1050^\circ) \\ &= 128(\cos 330^\circ + i \sin 330^\circ) \\ &= 128\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\ &= 64\sqrt{3} - 64i.\end{aligned}$$

Exercise 61

Perform the indicated operations and express results in algebraic form.

1. $4(\cos 310^\circ + i \sin 310^\circ) \cdot 7(\cos 140^\circ + i \sin 140^\circ)$.

2. $8(\cos 170^\circ + i \sin 170^\circ) \cdot (\cos 190^\circ + i \sin 190^\circ)$.

3. $3(\cos 100^\circ + i \sin 100^\circ) \cdot 5(\cos 80^\circ + i \sin 80^\circ)$.

4. $6(\cos 160^\circ + i \sin 160^\circ) \cdot 6(\cos 110^\circ + i \sin 110^\circ)$.

5. $5(\cos 25^\circ + i \sin 25^\circ) \cdot 2(\cos 100^\circ + i \sin 100^\circ)$.

6. $2(\cos 20^\circ + i \sin 20^\circ) \cdot 4(\cos 40^\circ + i \sin 40^\circ)$.

7. $\dfrac{6(\cos 320^\circ + i \sin 320^\circ)}{2(\cos 20^\circ + i \sin 20^\circ)}$.

8. $\dfrac{12(\cos 190^\circ + i \sin 190^\circ)}{3(\cos 10^\circ + i \sin 10^\circ)}$.

9. $\dfrac{8(\cos 340^\circ + i \sin 340^\circ)}{4(\cos 100^\circ + i \sin 100^\circ)}$.

10. $\dfrac{14(\cos 99^\circ + i \sin 99^\circ)}{7(\cos 9^\circ + i \sin 9^\circ)}$.

11. $\dfrac{-10\sqrt{2} - 10i\sqrt{2}}{5(\cos 75^\circ + i \sin 75^\circ)}$.

12. $\dfrac{-\sqrt{3} + i}{\cos 15^\circ + i \sin 15^\circ}$.

For each of the following products (a) express the factors in trigonometric form, (b) find the product trigonometrically, (c) check your result by finding the product algebraically.

13. $(3 - 3i)i$.

14. $(5 + 5i)(1 - i)$.

15. $(4\sqrt{3} - 4i)(1 + i\sqrt{3})$.

16. $-2i(\sqrt{3} + i)$.

Use De Moivre's theorem to find the indicated powers.

17. $[2(\cos 20^\circ + i \sin 20^\circ)]^3$.

18. $[\sqrt[5]{6}(\cos 54^\circ + i \sin 54^\circ)]^{10}$.

19. $[\sqrt{5}(\cos 105^\circ + i \sin 105^\circ)]^6$.

20. $[3(\cos 5^\circ + i \sin 5^\circ)]^4$.

21. $(-1 - i)^5$.

22. $(-3 + 3i)^4$.

23. $(2 + 2i)^6$.

24. $(1 - i)^7$.

25. $(3 - i\sqrt{3})^4$.

26. $\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{15}$.

27. $\left(\frac{-1 - i\sqrt{3}}{2}\right)^{10}$.

28. $(-\sqrt{3} + i)^5$.

29. $(2 + i)^6$.

30. $(3 - 4i)^4$.

31. $(-1 + 3i)^4$.

32. Prove that the reciprocal of $\cos\theta + i\sin\theta$ is

$$\cos(360° - \theta) + i\sin(360° - \theta).$$

117. Roots of complex numbers. *Theorem. The* **n**th *roots of* $r(\cos\theta + i\sin\theta)$ *are given by the formula*

$$\sqrt[n]{r}\left[\cos\frac{\theta + k\cdot 360°}{n} + i\sin\frac{\theta + k\cdot 360°}{n}\right],$$

where $k = 0, 1, 2, \cdots, n - 1$.

Proof. Assuming De Moivre's theorem is true when n is the reciprocal of a positive integer, we have

$$\begin{aligned}\sqrt[n]{r(\cos\theta + i\sin\theta)} &= [r(\cos\theta + i\sin\theta)]^{\frac{1}{n}}\\ &= r^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right).\end{aligned}$$

Since $\cos\theta$ and $\sin\theta$ are periodic functions with a period of 360°, we can say that $\cos\theta = \cos(\theta + k\cdot 360°)$ and $\sin\theta = \sin(\theta + k\cdot 360°)$. Hence

$$\sqrt[n]{r(\cos\theta + i\sin\theta)} = \sqrt[n]{r}\left(\cos\frac{\theta + k\cdot 360°}{n} + i\sin\frac{\theta + k\cdot 360°}{n}\right).$$

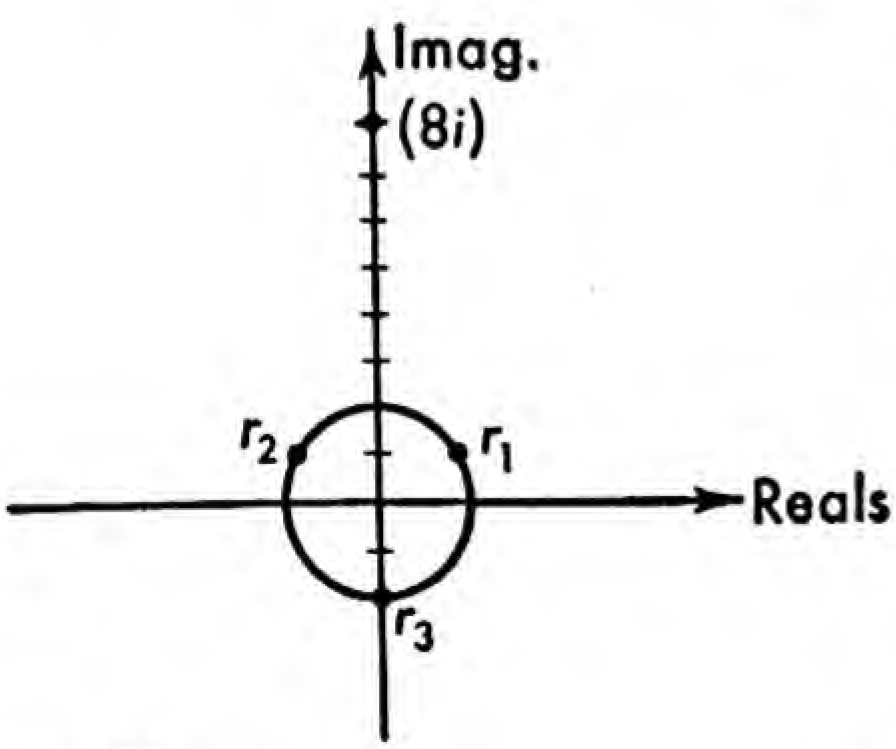


FIG. 28

It is easy to show that the right side of this equation takes on n distinct values when k takes on the values $0, 1, 2, \cdots, n - 1$. But if k takes on a value larger than $(n - 1)$, the result is merely a duplication of one of the n roots already found.

Example 1. Find the three cube roots of $8i$.

Solution. Plot the number $8i$ and then put it in trigonometric form:

$$8i = 8(\cos 90^\circ + i \sin 90^\circ).$$

Apply the theorem on roots. The three cube roots of $8i$ are

$$\sqrt[3]{8}\left(\cos \frac{90^\circ + k \cdot 360^\circ}{3} + i \sin \frac{90^\circ + k \cdot 360^\circ}{3}\right)$$
$$= 2[\cos (30^\circ + k \cdot 120^\circ) + i \sin (30^\circ + k \cdot 120^\circ)].$$

Let the three roots be r_1, r_2, r_3. Then

$$r_1 = 2(\cos 30^\circ + i \sin 30^\circ) = \sqrt{3} + i \qquad [k = 0]$$
$$r_2 = 2(\cos 150^\circ + i \sin 150^\circ) = -\sqrt{3} + i \qquad [k = 1]$$
$$r_3 = 2(\cos 270^\circ + i \sin 270^\circ) = -2i. \qquad [k = 2]$$

The three roots are equally spaced on a circle of radius 2 and center at the origin (Fig. 28). Notice that for $k = 3$, we obtain r_1 again.

Exercise 62

Find all the indicated roots of the following complex numbers. Write results in trigonometric form and also in algebraic form. Use a three-place table of trigonometric functions (Table II).

1. The fourth roots of $16(\cos 200^\circ + i \sin 200^\circ)$.
2. The fifth roots of $32(\cos 305^\circ + i \sin 305^\circ)$.
3. The sixth roots of $\cos 132^\circ + i \sin 132^\circ$.
4. The square roots of $36(\cos 36^\circ + i \sin 36^\circ)$.
5. The square roots of $-64i$.
6. The square roots of $2i$.
7. The square roots of $8 - 8i\sqrt{3}$.
8. The cube roots of $-8i$.
9. The cube roots of $4\sqrt{2} + 4i\sqrt{2}$.
10. The cube roots of $-\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}$.
11. The cube roots of -1000.
12. The fourth roots of 81.
13. The fourth roots of $128 - 128i\sqrt{3}$.
14. The fourth roots of -256.

15. The fourth roots of $-8-8i\sqrt{3}$.

16. The fifth roots of $-2\sqrt{31}+30i$.

17. The square roots of $-7+24i$.

18. The cube roots of $3+i\sqrt{55}$.

Find all the roots of the following equations.

19. $x^5-1=0$.

Hint. The roots of the equation $x^5-1=0$ are the five fifth roots of 1.

20. $x^3=27i$.

21. $x^6=-64$.

22. $x^2=80+60i$.

23. $x^2=-5+11i$.

chapter 16 Theory of equations

118. Integral rational equations; polynomials. An **integral rational equation** in x is an equation that can be written in the form

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0,$$

where n is a positive integer, and the a's are constants with $a_0 \neq 0$. The left side of this equation,

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

is called a **polynomial** of degree n in x, or an **integral rational function** of degree n.

Illustration. $7x^3 - 8x^2 + 9 = 0$ is an integral rational equation with $n = 3$, $a_0 = 7$, $a_1 = -8$, $a_2 = 0$, $a_3 = 9$.

In this chapter we shall study methods of solving integral rational equations. Before this can be done, we must investigate certain properties of polynomials and equations. These properties are exhibited in the theorems that follow.

119. Remainder theorem (theorem 1). *If a polynomial $f(x)$ is divided by $(x - r)$ until a constant remainder is obtained, then this remainder is equal to $f(r)$.*

Proof. Let r be any constant. In the division of $f(x)$ by $(x - r)$, let $q(x)$ be the quotient and let R be the constant remainder.

For all cases of division (Art. 11, equation 2), we have the following identity:

$$\textit{dividend} \equiv (\textit{divisor})(\textit{quotient}) + \textit{remainder}.$$

Hence $$f(x) \equiv (x - r)q(x) + R.$$

Since this is an identity, it must be true for all values of x, including $x = r$, for which we have

$$f(r) = (r - r)q(r) + R = 0 \cdot q(r) + R.$$

Hence $$f(r) = R.$$

Example 1. Verify the remainder theorem for

$$f(x) = 5x^3 - 6x^2 - 7x + 9 \quad \text{and} \quad r = 2.$$

Solution. We must show that when $f(x)$ is divided by $x - 2$, the remainder is $f(2)$.

$$\begin{array}{rrrrrl}
 & 5x^2 & + 4x & + 1 & = q(x) & \\
f(x) = & 5x^3 & - 6x^2 & - 7x & + 9 & \big|\ x - 2 \\
 & 5x^3 & - 10x^2 & & & \\
\hline
 & & 4x^2 & - 7x & & \\
 & & 4x^2 & - 8x & & \\
\hline
 & & & x & + 9 & \\
 & & & x & - 2 & \\
\hline
 & & & & 11 & = R
\end{array}$$

But
$$\begin{aligned} f(2) &= 5 \cdot 2^3 - 6 \cdot 2^2 - 7 \cdot 2 + 9 \\ &= 40 - 24 - 14 + 9 \\ &= 11. \end{aligned}$$

It is to be noted that this division enables us to write the following identity:

$$5x^3 - 6x^2 - 7x + 9 = (x - 2)(5x^2 + 4x + 1) + 11.$$

120. Factor theorem (theorem 2).

If r is a root of $f(x) = 0$,
then $(x - r)$ is a factor of $f(x)$.

Proof. Since r is a root of $f(x) = 0$, it follows that $f(r) = 0$. But the remainder theorem says that $f(r) = R$. Hence $R = 0$ and

$$f(x) = (x - r)q(x) + 0.$$

Therefore $(x - r)$ is a factor of $f(x)$. What is the other factor?

Illustration 1. By substitution, we see that 1 is a root of

$$x^3 + 8x^2 - 5x - 4 = 0.$$

The factor theorem says that $(x - 1)$ must be a factor of the left side of the equation. The student should verify by division that

$$x^3 + 8x^2 - 5x - 4 = (x - 1)(x^2 + 9x + 4).$$

121. Converse of the factor theorem.

$$\begin{array}{llll} \textit{If} & (x - r) & \textit{is a factor} & \textit{of } f(x), \\ \textit{then} & r & \textit{is a root} & \textit{of } f(x) = 0. \end{array}$$

Proof. By hypothesis, if $f(x)$ is divided by $(x - r)$, the division is exact, i.e., the remainder is 0:

$$f(x) \equiv (x - r)q(x).$$

Set $x = r$: $\quad f(r) = (r - r)q(r) = 0 \cdot q(r) = 0.$

Since $f(r) = 0$, it follows that r is a root of $f(x) = 0$.

This theorem has been used in solving quadratic equations by factoring.

122. Synthetic division. In finding the roots of the equation $f(x) = 0$, it will be necessary to divide $f(x)$ by $(x - 1)$, $(x - 2)$, $(x + 1)$, etc. Instead of doing this by long division, we shall employ a much shorter method called **synthetic division.**

Illustration 1. We shall exhibit the division of $(4x^3 - 13x^2 + 16x + 11)$ by $(x - 2)$ using (A) ordinary long division, (B) and (C) abridgments of (A), and (D) synthetic division.

(A)

Long Division

$$\begin{array}{rrrrl}
\mathbf{4x^2} & \mathbf{-\ 5x} & \mathbf{+\ 6} & & = \text{quotient} \\
\hline
\mathbf{4x^3} & -\ 13x^2 & +\ 16x & +\ 11 & \big|\ \underline{x - 2} \\
4x^3 & -\ 8x^2 & & & \\
\hline
 & \mathbf{-\ 5x^2} & +\ 16x & & \\
 & -\ 5x^2 & +\ 10x & & \\
\hline
 & & \mathbf{6x} & +\ 11 & \\
 & & 6x & -\ 12 & \\
\hline
 & & & \text{Remainder} = \mathbf{23} &
\end{array}$$

(B)

$$\begin{array}{rrrrl}
\mathbf{4x^2} & \mathbf{-\ 5x} & \mathbf{+\ 6} & & \\
\hline
\mathbf{4x^3} & -\ 13x^2 & +\ 16x & +\ 11 & \big|\ \underline{x - 2} \\
 & -\ 8x^2 & +\ 10x & -\ 12 & \\
\hline
 & \mathbf{-\ 5x^2} & +\ \mathbf{6x} & +\ \mathbf{23} &
\end{array}$$

(C)

$$\begin{array}{rrrrl}
4 & -\ 13 & +\ 16 & +\ 11 & \big|\ \underline{1 - 2} \\
 & -\ 8 & +\ 10 & -\ 12 & \\
\hline
\mathbf{4} & \mathbf{-\ 5} & \mathbf{+\ 6} & \mathbf{+\ 23} &
\end{array}$$

The coefficients of the quotient are printed in boldface in the division details to show where they originate. Each encircled term in (A) is superfluous because it duplicates the term above it. In

(B) these terms have been deleted and the remaining terms have been moved up into more compact form. In (C) only the coefficients are written; the quotient has been omitted because its coefficients appear on the bottom line (if we move the first coefficient down to this line).

Since we are dividing by $(x - r)$, the coefficient of x is 1 and need not appear in the division. Instead of using -2 and subtracting, we shall use $+2$ and *add*.

(D)

Synthetic Division of $(4x^3 - 13x^2 + 16x + 11)$ by $(x - 2)$.

$$\begin{array}{r|l} 4 - 13 + 16 + 11 & \underline{2} \\ + \ 8 - 10 + 12 & \\ \hline 4 - \ 5 + \ 6 + 23 & \end{array}$$

Quotient $= 4x^2 - 5x + 6$; remainder $= 23 = f(2)$.

Notice that the 4 is brought down and then multiplied by 2 to get the 8. We add -13 to $+8$ and get -5, etc.

Rule for synthetic division. *To divide $f(x)$ by $(x - r)$:*

1. *Write $f(x)$ in descending powers of x; place the coefficients on the first line* (supply zero as the coefficient of each missing power); *write r at the right.*

2. *Bring the first coefficient down to the first place in the third line and multiply it by r; write the product on the second line under the second coefficient of $f(x)$; add and write the sum on the third line. Multiply this by r and write the product on the second line under the next coefficient of $f(x)$ and add. Continue this process.*

3. *The last number in the third line is the remainder; the other numbers, reading from left to right are the coefficients of the quotient, arranged in descending powers of x.*

Example 1. Divide $2x^4 + 11x^3 - 33x + 44$ by $x + 3$.

Solution. Here $x - r = x + 3$; hence $-r = 3$, and $r = -3$. Since the x^2 term is missing, we supply a zero coefficient.

$$\begin{array}{r|l} 2 + 11 + \ 0 - 33 + 44 & \underline{-3} \\ - \ 6 - 15 + 45 - 36 & \\ \hline 2 + \ 5 - 15 + 12 + \ 8 & \end{array}$$

Quotient $= 2x^3 + 5x^2 - 15x + 12$; remainder $= 8$.

Remembering that

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}},$$

we have

$$\frac{2x^4 + 11x^3 - 33x + 44}{x + 3} = 2x^3 + 5x^2 - 15x + 12 + \frac{8}{x + 3}.$$

The remainder theorem says that $f(r)$ is equal to the remainder when $f(x)$ is divided by $(x - r)$. Hence $f(r)$ can be obtained by synthetic division with the number r placed at the right.

Example 2. If $f(x) = x^3 + 6x^2 - 4x - 9$, find $f(2)$, $f(-1)$.

Solution.

$$\begin{array}{rrrr|l} 1 + & 6 - & 4 - & 9 & \underline{2} \\ & +2 + & 16 + & 24 & \\ \hline 1 + & 8 + & 12 + & 15 & \qquad f(2) = 15. \end{array}$$

$$\begin{array}{rrrr|l} 1 + & 6 - & 4 - & 9 & \underline{-1} \\ & -1 - & 5 + & 9 & \\ \hline 1 + & 5 - & 9 + & 0 & \qquad f(-1) = 0. \end{array}$$

Since $f(-1) = 0$, we see that $f(x)$ is *exactly divisible* by $(x + 1)$:

$$x^3 + 6x^2 - 4x - 9 = (x + 1)(x^2 + 5x - 9).$$

Exercise 63

Find the quotient and remainder by using synthetic division. Check by use of long division.

1. $(3x^4 - 11x^3 - 22x^2 + 9) \div (x - 5)$.
2. $(5x^4 - 17x^2 - 3) \div (x + 2)$.

Find the quotient and remainder by using synthetic division.

3. $(x^3 - 7x^2 + 9) \div (x - 1)$.
4. $(x^3 - 5x - 2) \div (x - 3)$.
5. $(2x^3 + 5x^2 - 7x + 20) \div (x + 4)$.
6. $(3x^3 - 8x^2 + 4x + 7) \div (x - 2)$.
7. $(3x^4 + 5x^3 + 6x^2 - 9x - 1) \div (x + 1)$.
8. $(2x^4 - x^3 - 19x^2 - 18) \div (x + 3)$.

9. $(8x^3 - 4x^2 - 6x - 7) \div (x - \frac{1}{2})$.

10. $(6x^4 + x^3 - 2x^2 + 9x - 5) \div (x + \frac{2}{3})$.

11. $(x^3 - .1x^2 + 1.2x + .05) \div (x + .7)$.

12. $(x^3 + .8x^2 - .16x - .148) \div (x - .3)$.

Use synthetic division and the remainder theorem in the following problems.

13. Given $f(x) = x^4 - x^2 - 5x - 7$; find $f(2), f(-3)$.

14. Given $f(x) = x^4 + 2x^3 + 3x - 4$; find $f(4), f(-1)$.

15. Given $g(x) = 2x^5 + 3x^4 - 4x^2 - 5$; find $g(1)$, $g(-2)$.

16. Given $g(x) = 3x^5 + 2x^4 - 4x^3 - 8x^2 + 6x + 7$; find $g(\frac{1}{3})$, $g(-4)$.

Use the factor theorem in the following problems.

17. Prove that $(x - 1)$ is a factor of $x^{100} - 1$.

18. Prove that $(x + 1)$ is a factor of $x^{99} + 1$.

19. Prove that $(x + 2)$ is a factor of $x^9 + 512$.

20. Prove that $(x - 2)$ is a factor of $x^8 - 256$.

21. Use synthetic division to show that $(x + 1)^2$ is a factor of $x^4 + 5x^3 + 14x^2 + 17x + 7$.

22. Use synthetic division to show that $(x - 2)^2$ is a factor of $5x^3 - 21x^2 + 24x - 4$.

123. Graphs of polynomials. In order to graph the polynomial $f(x)$, we first assign values to x and then compute corresponding values of $f(x)$. This computation can be done by using synthetic division. See Art. 35 for suggested procedure in selecting values to be assigned to x.

Example 1. Graph: $f(x) = 2x^3 - 5x^2 - 12x + 8$.

Solution. In the following table, the values of $f(x)$ were obtained by synthetic division. For example, $f(2)$ was computed as follows:

$$\begin{array}{rrrr|l} 2 & -\ 5 & -\ 12 & +\ 8 & 2 \\ & +\ 4 & -\ 2 & -\ 28 & \\ \hline 2 & -\ 1 & -\ 14 & -\ 20 = f(2). & \end{array}$$

x		-2	-1	0	1	2	3	4	
$f(x)$		-4	13	8	-7	-20	-19	8	

The graph of this polynomial of degree *three* (Fig. 29) shows that the curve meets the x-axis in *three* points: A, B, and C.

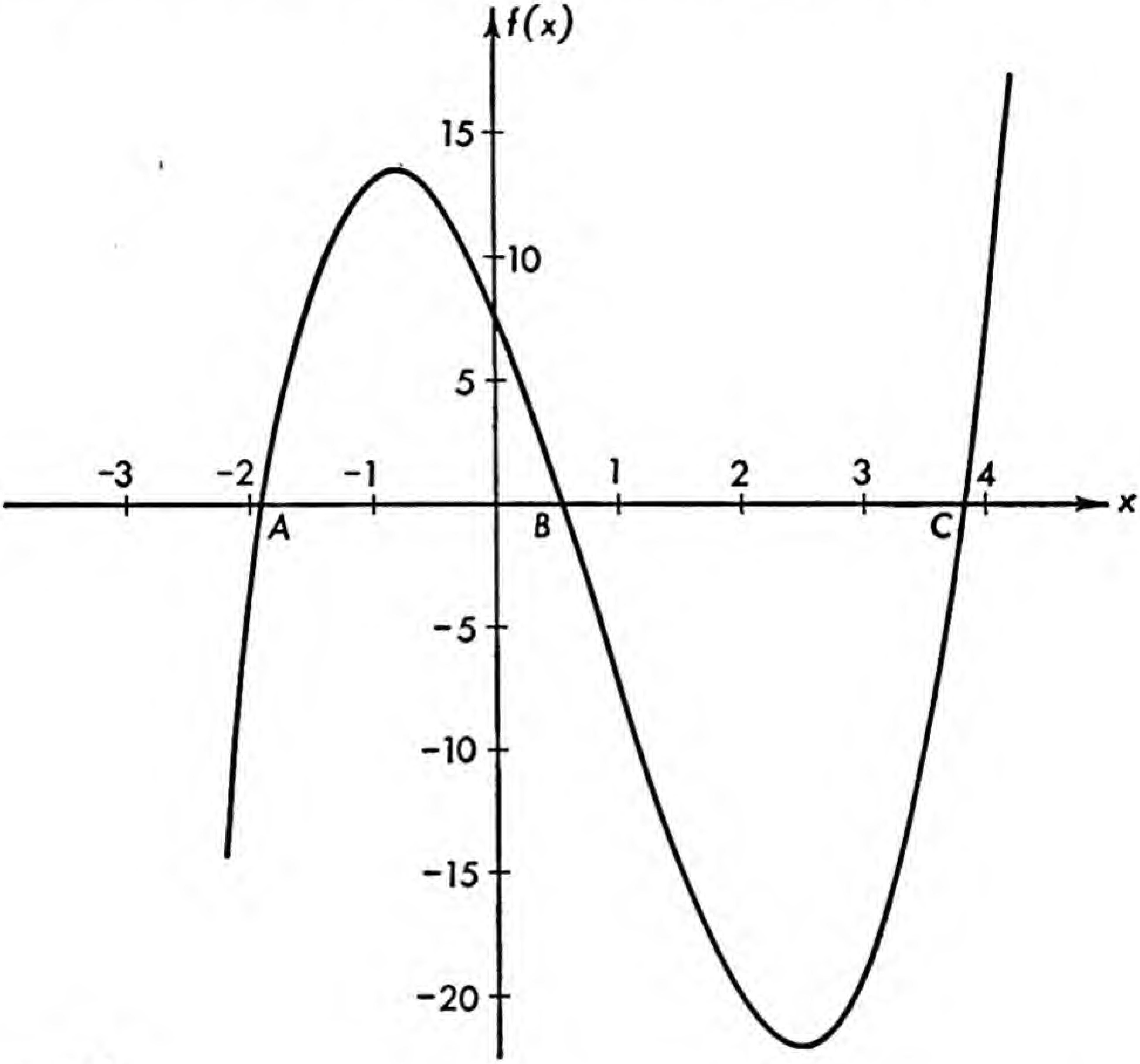


FIG. 29

Every polynomial $f(x)$ is said to be **single-valued** because for each value assigned to x there corresponds one and only one value of $f(x)$. The polynomial $f(x)$ is said to be **continuous** because its graph is a smooth curve without breaks, i.e., it does not consist of disconnected portions. *Every polynomial is a single-valued, continuous function.*

In graphing polynomials, it is useful to remember the following facts.

I. The graph of a polynomial of degree n meets the x-axis (or any other line) in *at most* n points.

II. If $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, then for sufficiently large values of x the sign of $f(x)$ will be the same as that of a_0x^n. For example, if $f(x) = 2x^3 - 13x^2 - 65x - 50$, then for sufficiently large positive values of x (in this case for $x > 10$), $f(x)$ will be positive. For sufficiently large negative values of x (in this case for $x \leqq -3$), $f(x)$ will be negative. This means that the graph of $f(x)$ meets the x-axis at points which lie in the range $-3 < x \leqq 10$.

124. Graphic solution of equations. The *real* roots of the equation $f(x) = 0$ are the x-coordinates of the points at which the graph of the polynomial $f(x)$ meets the x-axis.

Example 1. Solve graphically: $2x^3 + 8 = 5x^2 + 12x$.

Solution. Transpose all terms to the left side of the equation:

$$2x^3 - 5x^2 - 12x + 8 = 0. \tag{1}$$

Let $f(x)$ represent the left side:

$$f(x) = 2x^3 - 5x^2 - 12x + 8. \tag{2}$$

The real roots of (1) are the x-coordinates of the points at which the graph of (2) meets the x-axis (since (2) reduces to (1) when $f(x) = 0$). From the graph of (2) in Fig. 29, we estimate the x's of A, B, C to be -1.9, .6, 3.8. Hence the roots of the given equation are approximately

$$-1.9, \quad .6, \quad 3.8.$$

Since these values were read from the curve, they are to be treated as approximations. If the student should read 3.7 or 3.9 instead of 3.8, his result would be acceptable.

In Art. 134 and Art. 136 we shall discuss methods of obtaining better approximations for the roots.

Exercise 64

Graph the following polynomials.

1. $x^3 - 4x + 2$.

2. $x^3 - 6x + 4$.

3. $-x^3 + 5x - 1$.

4. $x^3 + x^2 - 13$.

5. $x^3 - x^2 - 11$.

6. $x^3 + x$.

7. $x^4 - 6x^2 + 10$.

8. $-x^4 + 8x^3 - 14x^2 - 8x + 15$.

Solve graphically.

9. $x^3 - 5x^2 + 2x + 8 = 0.$

10. $x^3 - 4x^2 - x + 4 = 0.$

11. $x^3 + x^2 - 5x + 2 = 0.$

12. $2x^3 + x^2 + 4 = 0.$

13. $x^4 - 21x + 8 = 0.$

14. $x^4 - 4x^3 - 2x^2 + 16x - 8 = 0.$

15. $x^4 - 2x^3 - 5x^2 + 6x = 0.$

16. $x^5 - 10x^3 + 9x = 0.$

125. Theorem on the number of roots of an equation (theorem 3). *Every integral rational equation of degree n has exactly n roots, provided a root of multiplicity m is counted as m roots.*

Illustration. The equation

$$2x^6 + 15x^5 + 3x^4 - 94x^3 + 84x^2 + 15x - 25 = 0,$$

of degree 6, has the roots $-5, -5, -\frac{1}{2}, 1, 1, 1$. We say that -5 is a root of multiplicity 2; it is called a **double root.** It is conventional to speak of $-\frac{1}{2}$ as a **single** (or simple) root. We call 1 a **triple root,** i.e., a root of multiplicity 3. If each root is counted as many times as it occurs, it is evident that the equation of degree 6 has exactly 6 roots. In factored form, the equation is

$$(x + 5)^2(2x + 1)(x - 1)^3 = 0.$$

The proof of theorem 3 depends on the **fundamental theorem of algebra**: *Every integral rational equation has at least one root.* We shall assume this to be true because the proof is beyond the scope of this book.*

Proof of theorem 3: Let $f_n(x)$ represent a polynomial of degree n:

$$f_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n.$$

By the fundamental theorem, the equation $f_n(x) = 0$ has at least one root, which we call r_1. The factor theorem says that $(x - r_1)$ must be a factor of $f_n(x)$. Hence

$$f_n(x) = (x - r_1)f_{n-1}(x),$$

where $f_{n-1}(x)$ is a polynomial of degree $n - 1$, whose term of highest degree is a_0x^{n-1}.

* For a proof see Dickson's *First Course in the Theory of Equations*, p. 155.

Now consider the equation $f_{n-1}(x) = 0$. By the fundamental theorem, it has a root r_2. By the factor theorem, $(x - r_2)$ is a factor of $f_{n-1}(x)$. Hence

$$f_{n-1}(x) = (x - r_2)f_{n-2}(x)$$

or

$$f_n(x) = (x - r_1)(x - r_2)f_{n-2}(x),$$

where $f_{n-2}(x)$ is a polynomial of degree $n - 2$, whose term of highest degree is a_0x^{n-2}.

Continuing this process, we find, after n steps,

$$f_n(x) = (x - r_1)(x - r_2) \cdots (x - r_n)f_0(x),$$

where $f_0(x)$ is a polynomial of degree 0, whose term of highest degree is a_0x^0. Hence $f_0(x)$ is the constant a_0. Therefore

$$f_n(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n), \tag{1}$$

and the roots of $f_n(x) = 0$ are $r_1, r_2, \cdots, r_n$.

We shall now prove that $f_n(x) = 0$ can have no more than n roots. Let t be any number different from $r_1, r_2, \cdots, r_n$. Substituting in (1), we have

$$f_n(t) = a_0(t - r_1)(t - r_2) \cdots (t - r_n).$$

Since none of the factors on the right side is zero, we see that $f_n(t) \neq 0$, and t cannot be a root of $f_n(x) = 0$.

Corollary. *If two polynomials in x, each of degree not greater than n, are equal for more than n distinct values of x, then the polynomials are identical, i.e., the coefficients of like powers are equal.*

Proof. Let the polynomials be

$$a_0x^n + a_1x^{n-1} + \cdots + a_n$$

and

$$b_0x^n + b_1x^{n-1} + \cdots + b_n.$$

By hypothesis, there are more than n distinct values of x for which

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = b_0x^n + b_1x^{n-1} + \cdots + b_n.$$

This means that the equation

$$(a_0 - b_0)x^n + (a_1 - b_1)x^{n-1} + \cdots + (a_n - b_n) = 0 \tag{2}$$

has more than n roots. Now suppose that one (or more) of the coefficients $(a_0 - b_0), (a_1 - b_1), \cdots, (a_n - b_n)$ is different from zero.

We would then have an equation of degree n, or less, having more than n distinct roots. By theorem 3, this is impossible. Therefore all the coefficients in (2) must be zero, and

$$a_0 = b_0,\ a_1 = b_1,\ \cdots,\ a_n = b_n.$$

This means that the polynomials are identical; they are equal for all values of x.

126. Theorem on imaginary roots (theorem 4). *If an imaginary number $(a + bi)$ is a root of an integral rational equation with real coefficients, then the conjugate imaginary number $(a - bi)$ is also a root.* This means that imaginary roots occur in pairs.

Proof. Let $(a + bi)$ be a root of the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \tag{1}$$

where $a_0,\ a_1,\ \cdots,\ a_{n-1},\ a_n$ are real numbers. Since $(a + bi)$ is a root, the left side of (1) must reduce to zero when $(a + bi)$ is substituted for x:

$$a_0(a + bi)^n + a_1(a + bi)^{n-1} + \cdots + a_{n-1}(a + bi) + a_n = 0. \tag{2}$$

Expanding and collecting terms, we find that (2) can be written in the form

$$R + Si = 0, \tag{3}$$

where R and S are real numbers. Using Art. 110, we see that (3) implies that

$$R = 0 \quad \text{and} \quad S = 0.$$

If $(a - bi)$ is substituted for x in (1), the result will be the same as in (2) and (3) except that each i is replaced with $-i$. The resulting equation is

$$R - Si = 0, \tag{4}$$

where R and S represent the same numbers as in (3).* Since $R = 0$ and $S = 0$, we conclude that $(a - bi)$ satisfies (1) and must be a root of it.

Illustration. Consider the equation

$$x^3 - 2x + 4 = 0.$$

* This would not necessarily be true if $a_0,\ a_1,\ \cdots,\ a_n$ were not real numbers, e.g., consider the equation $ix + 1 = 0$.

Substitute $(1+i)$ for x: $1+3i-3-i-2-2i+4=0$.

Collect real and imaginary terms:

$$(1-3-2+4)+(3-1-2)i=0$$
$$0+0\cdot i=0.$$

Hence $(1+i)$ is a root of the given equation.

When $(1-i)$ is substituted for x in the original equation, we have

$$1-3i-3+i-2+2i+4=0$$
$$(1-3-2+4)-(3-1-2)i=0$$
$$0-0\cdot i=0.$$

Hence $(1-i)$ is also a root of the equation.

In this case $R=1-3-2+4=0$ and $S=3-1-2=0$.

Using a method of proof very much like that for theorem 3, we can establish the following

Theorem. If an equation with *rational coefficients* has the irrational root $a+\sqrt{b}$, where a and b are rational numbers,* then the equation also has the root $a-\sqrt{b}$.

127. Formation of an equation when the roots are given. If the roots of an equation are $r_1, r_2, \cdots, r_n$, the equation can be written in the form

$$(x-r_1)(x-r_2)\cdots(x-r_n)=0.$$

This statement is a direct consequence of the factor theorem.

Example 1. Form an equation with integral coefficients having the roots $3+2i, \frac{4}{5}, -1, -1$.

Solution. Since the coefficients are to be integers, they must be real numbers. Hence theorem 4 applies, and $3-2i$ must also be a root. The equation is

$$(x-[3+2i])(x-[3-2i])(x-\tfrac{4}{5})(x-[-1])^2=0$$
$$(\overline{x-3}-2i)(\overline{x-3}+2i)(5x-4)\dagger(x^2+2x+1)=0$$
$$(x^2-6x+13)(5x^3+6x^2-3x-4)=0$$
$$5x^5-24x^4+26x^3+92x^2-15x-52=0.$$

* Of course b cannot be a perfect square or else $\sqrt{b}$ would be rational and $a+\sqrt{b}$ would be rational.

† Multiply both sides of the equation by 5 to clear of fractions.

If the left side of an equation $f(x) = 0$ is expressed as the product of linear and quadratic factors, then the roots of the equation can be found by applying the converse of the factor theorem.

Example 2. Solve: $x(x - 7)^3(x^2 + 8x + 97) = 0$.

Solution. Setting each factor equal to zero and then solving, we find the roots to be

$$0, \quad 7, \quad 7, \quad 7, \quad -4 + 9i, \quad -4 - 9i.$$

Notice that 7 is a triple root. The two conjugate imaginary roots were found by solving $x^2 + 8x + 97 = 0$ by completing the square.

We shall use the following notation to indicate the roots.

$$r_1 = 0, \quad r_2 = 7, \quad r_3 = 7, \quad r_4 = 7, \quad r_5 = -4 + 9i, \quad r_6 = -4 - 9i.$$

Exercise 65

Solve.

1. $(x + 5)^2(x - 1)(x^2 - 6x + 2) = 0$.

2. $(7x - 3)^2(x + 4)(x^2 + 25) = 0$.

3. $(2x - 1)^3(x + 8)(x^2 + 10x + 29) = 0$.

4. $(3x^2 - 7x)(x^2 - 12x + 3) = 0$.

5. $(9x^2 + 4)x^2(x^2 - 7) = 0$.

6. $(x + 9)^3(x - 6)^2(2x^2 - 6x + 1) = 0$.

Form an equation with integral coefficients having only the given numbers as roots.

7. $3, 3, 0, -1$.

8. $2, -5, -5$.

9. $1, \frac{3}{4}, -\frac{1}{2}$.

10. $-1, 0, 0, \frac{1}{3}, -\frac{2}{7}$.

11. $-2, \sqrt{11}, -\sqrt{11}$.

12. $\frac{1}{3}, \sqrt{10}, -\sqrt{10}$.

13. $3, i, -i$.

14. $-9, 2i, -2i$.

15. $1, \frac{1}{2} + 3i, \frac{1}{2} - 3i$.

16. $-3, 5 + \frac{1}{2}i, 5 - \frac{1}{2}i$.

17. $-7, 0, 0, 2 + \sqrt{6}, 2 - \sqrt{6}$.

18. $1, \frac{1}{2}(3 + \sqrt{5}), \frac{1}{2}(3 - \sqrt{5})$.

19. 1 as a double root and 0 as a triple root.

20. 5 as a double root and $-\frac{1}{5}$ as a single root.

21. Form a third-degree equation with real coefficients having 6 and $(-1 + i)$ as roots.

22. Form a fourth-degree equation with real coefficients having i and $(3 - i)$ as roots.

23. Prove that a third-degree equation with real coefficients has three real roots or else one real and two imaginary roots.

24. Prove that every equation of odd degree with real coefficients must have at least one real root.

25. Use synthetic division to show that

$$x^4 - 8x^3 + 16x^2 + 12x - 45 = 0$$

has 3 as a double root. Find all the roots.

26. Use synthetic division to show that

$$x^4 - 4x^3 - x^2 + 14x + 10 = 0$$

has -1 as a double root. Find all the roots.

27. Use synthetic division to show that

$$4x^5 - 12x^4 + 7x^3 + 11x^2 - 15x + 5 = 0$$

has 1 as a triple root. Find all the roots.

28. Use synthetic division to show that

$$9x^4 - 48x^3 + 43x^2 - 4x - 4 = 0$$

has $\frac{2}{3}$ as a double root. Find all the roots.

* **128. Graph of a factored polynomial.** When a polynomial is expressed as the product of linear factors, its graph can be sketched quite rapidly. Let

$$f(x) = (x - r)^m g(x),$$

where $g(x)$ is a polynomial that does not contain $(x - r)$ as a factor. Then it can be shown that

I. If $m = 1$, then at $x = r$ the graph of $f(x)$ crosses the x-axis at an angle.

II. If m is odd but greater than 1, then at $x = r$ the curve is tangent to and crosses the x-axis.

III. If m is even, then at $x = r$ the curve is tangent to the x-axis but does not cross it.

Illustration. The graph (Fig. 30) of

$$f(x) = (x + 1)^4(7x - 5)(x - 2)^3$$

may be sketched by noticing that

* May be omitted.

1. The curve meets the x-axis at only three distinct points, for which $x = -1, \frac{5}{7}, 2$.

2. The curve tends to go up on the right, i.e., for sufficiently large values of x (in this case for $x > 2$), $f(x)$ is positive and increases as x increases.

3. At $x = 2$ there is a crossing with a tangency [because the exponent on $(x - 2)$ is 3]; at $x = \frac{5}{7}$ there is a crossing at an angle; at $x = -1$ there is a tangency but no crossing.

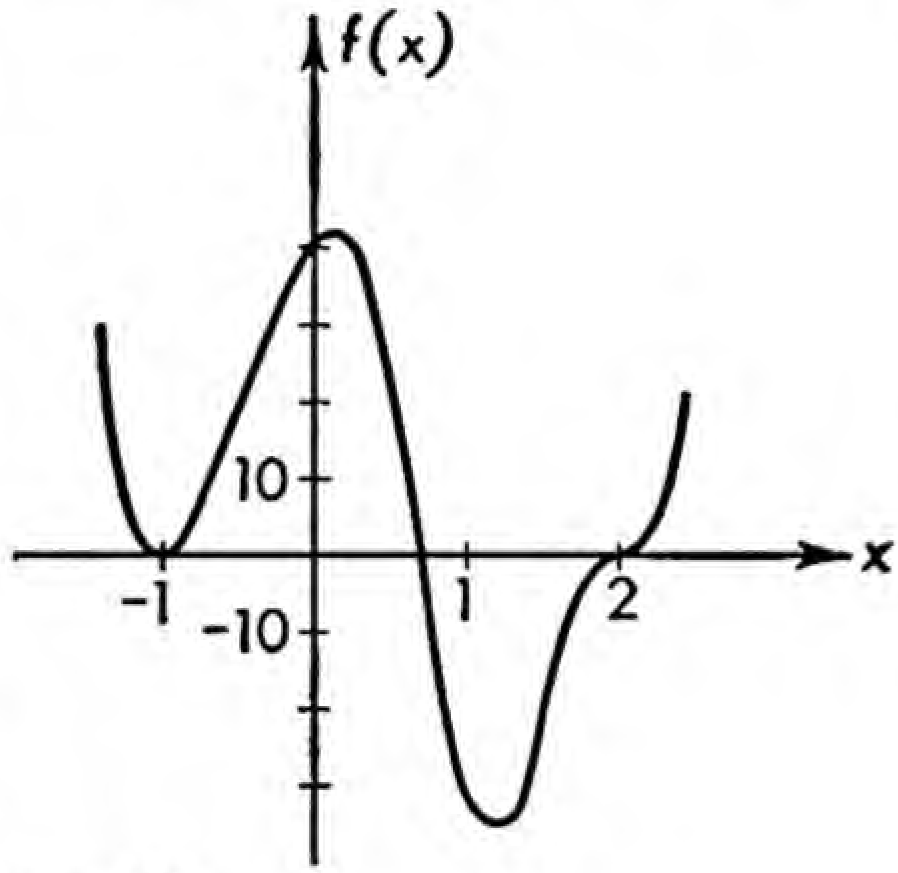


FIG. 30

In this article we are interested in the character of only that part of the curve that lies in the neighborhood of the x-axis. Consequently a fairly good check can be obtained by determining the sign of $f(x)$ for a few other values of x. For example, for $x = 0$, $f(x) = (+)^4(-)(-)^3 = +$. The following table was obtained in this way.

x	-2	-1	0	$\frac{5}{7}$	1	2	3
$f(x)$	$+$	0	$+$	0	$-$	0	$+$

The corresponding equation,

$$(x + 1)^4(7x - 5)(x - 2)^3 = 0$$

has $\frac{5}{7}$ as a single root, 2 as a triple root, and -1 as a root of multiplicity 4.

Exercise 66

Sketch graphs of the following polynomials. Indicate clearly all crossings and tangencies.

1. $(x + 1)(x - 3)^2$.

2. $(x - 1)^2(3x - 7)$.

3. $(x + 2)^3(x - 2)$.

4. $-5x(x - 2)^3$.

5. $(2x + 3)^2(x - 1)^3$.

6. $(x + 4)^3(x + 1)^2$.

7. $-(x-2)(x-4)^3(x-6)^2$.
8. $(x+3)(x-1)^2(x-5)^3$.
9. $(x+2)(x+1)^2(x-1)(x-2)^3$.
10. $(x-1)^4(x-2)^5(x-3)$.
11. $(x+4)^3(2x+5)^2(x-1)$.
12. $2(3x+1)(x-1)(x-2)$.

129. The roots of $f(-x) = 0$ are the negatives of the roots of $f(x) = 0$. To prove this, let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

By theorem 3, we know that $f(x) = 0$ has n roots which we will call $r_1, r_2, \cdots, r_n$. Then by (1) of Art. 125,

$$f(x) = a_0(x - r_1)(x - r_2) \cdots (x - r_n).$$

To get $f(-x)$, we replace x with $-x$:

$$f(-x) = a_0(-x - r_1)(-x - r_2) \cdots (-x - r_n).$$

The roots of $f(-x) = 0$ can be found by setting each of the factors of $f(-x)$ equal to zero. We find the roots of $f(-x) = 0$ are $-r_1, -r_2, \cdots, -r_n$.

Example 1. Find an equation whose roots are the negatives of the roots of $f(x) = 4x^4 - 15x^3 + 25x + 6 = 0$.

Solution. Replace x with $-x$:

$$f(-x) = 4(-x)^4 - 15(-x)^3 + 25(-x) + 6 = 0$$
$$4x^4 + 15x^3 - 25x + 6 = 0.$$

The roots of the given equation $f(x) = 0$ are $-1, -\frac{1}{4}, 2$, and 3. The roots of the new equation $f(-x) = 0$ are $1, \frac{1}{4}, -2$, and -3.

Notice that $f(-x)$ differs from $f(x)$ only in that the signs of the terms of *odd degree* have been changed.

130. Descartes' rule of signs. Let a polynomial $f(x)$ be arranged in descending powers of x. Two consecutive terms of $f(x)$ are said to have a variation of sign if their coefficients have different signs. Zero coefficients are to be disregarded.

Illustration. The polynomial

$$f(x) = 7x^6 - 8x^5 - 9x^2 + 6x - 1$$

has three variations of sign. The signs are $+ - - + -$. There are three variations as indicated by the arrows.

The polynomial

$$f(-x) = 7x^6 + 8x^5 - 9x^2 - 6x - 1$$

has one variation of sign.

Descartes' rule of signs (theorem 5). *The number of positive roots of $f(x) = 0$ is either equal to the number of variations of sign in $f(x)$, or else it is less than this number by an even integer. The number of negative roots of $f(x) = 0$ is either equal to the number of variations of sign in $f(-x)$, or else it is less than this number by an even integer.*

We shall use this theorem without proof. A general proof may be found in Dickson's *First Course in the Theory of Equations*, page 72.

Example 1. Apply Descartes' rule to the equation

$$5x^3 - 6x^2 - 7 = 0.$$

Solution. In this case,

$$f(x) = 5x^3 - 6x^2 - 7$$

has one variation of sign. Hence the given equation has exactly one positive root. (It could not have fewer positive roots because if 1 is diminished by an even integer, the result would be negative — an impossible number of positive roots.)

Moreover,

$$f(-x) = -5x^3 - 6x^2 - 7$$

has no variation of sign. Hence the given equation has no negative root.

Therefore the given equation, which is of degree 3, must have one positive root and two imaginary roots.

Example 2. Apply Descartes' rule to the equation

$$x^4 + 5x^3 + 6x^2 + 7x + 8 = 0.$$

Solution. Since $f(x)$ has no variation of sign, there is no positive root. (By inspection we see that if a positive number is substituted for x, all terms on the left side are positive, and their sum cannot be zero.)

Since $f(-x) = x^4 - 5x^3 + 6x^2 - 7x + 8$ has four variations of sign, the number of negative roots of the given equation is 4 or 2 or 0. The various possibilities are listed below:

POSITIVE	NEGATIVE	IMAGINARY
0	4	0
0	2	2
0	0	4

Exercise 67

Apply Descartes' rule of signs to the following equations. Make a list of the various possibilities.

1. $x^3 - 7x^2 + 8 = 0.$
2. $x^3 + 5x + 2 = 0.$
3. $7x^3 + x^2 = x + 6.$
4. $x^3 + x = x^2 + 1.$
5. $x^4 + x^3 + x^2 - 2x - 1 = 0.$
6. $5x^4 - x^3 - x^2 + 6x + 9 = 0.$
7. $x^4 + x^2 - 3 = 0.$
8. $x^4 - 6x^3 + 7x^2 - 8x + 9 = 0.$
9. $7x^6 + x^2 + 1 = 0.$
10. $x^4 - x^2 = 6 - 5x.$
11. $x^5 - 4x^3 - 6x - 2 = 0.$
12. $x^5 + x = 0.$
13. $x^7 + 3 = x^4 + x.$
14. $x^6 - x^5 + x^4 - 2 = 0.$
15. $x^4 + 5x^3 - 6x^2 = 0.$
16. $3x^8 + x^6 + x^2 - 5 = 0.$

131. Upper and lower limits for the roots. Any number u is called an **upper limit** for the real roots of an equation if each of the roots is less than or equal to u. Similarly, l is a lower limit if there is no root less than l.

Theorem on limits for the roots (theorem 6). *If k is positive and if all the numbers in the third line of the synthetic division of $f(x)$ by $(x - k)$ are positive or zero, then k is an upper limit for the real roots of $f(x) = 0$; if k is negative and if all numbers in the third line alternate in sign, then k is a lower limit for the roots.*

To prove the first part of the theorem, notice that for any number larger than k, all numbers (except the first) in the third line will be larger. Consequently the remainder, which is the last number, cannot be zero. Therefore no number larger than k can be a root.

The second part of the theorem can be proved by a similar method.

Example 1. Find upper and lower limits for the roots of the equation

$$2x^3 + x^2 - 16 = 0.$$

Solution. We shall first test 1 to see if it is an upper limit.

$$\begin{array}{rrrr|l} 2 & +1 & +0 & -16 & \underline{1} \\ & +2 & +3 & +3 & \\ \hline 2 & +3 & +3 & -13 & \end{array}$$

Hence 1 is not an upper limit.

We now test 2.

$$\begin{array}{rrrr|l} 2 & +1 & +0 & -16 & \underline{2} \\ & +4 & +10 & +20 & \\ \hline 2 & +5 & +10 & +4 & \end{array}$$

Since all numbers in the third line are positive, 2 is an upper limit.

In finding a lower limit, we try -1, -2, -3, etc.

$$\begin{array}{rrrr|l} 2 & +1 & +0 & -16 & \underline{-1} \\ & -2 & +1 & -1 & \\ \hline 2 & -1 & +1 & -17 & \end{array}$$

Since all numbers in the third line alternate in sign, -1 is a lower limit.

Therefore all real roots of the equation lie between -1 and 2.

Exercise 68

Find upper and lower limits for the roots of the following equations.

1. $3x^3 + x^2 - 5 = 0$.
2. $5x^3 - 2x^2 - x + 4 = 0$.
3. $2x^3 - 9x^2 + 5x + 7 = 0$.
4. $3x^3 + 7x^2 + 2x - 4 = 0$.
5. $5x^3 - 2x^2 + 8x - 36 = 0$.
6. $7x^3 + x^2 + 4x + 48 = 0$.
7. $x^3 - 3x^2 + x + 4 = 0$.
8. $3x^3 - x^2 - 13x - 8 = 0$.
9. $2x^4 + 2x^3 - 7x^2 + 11 = 0$.
10. $3x^4 - 4x^3 - 5x^2 - 6x + 7 = 0$.
11. $5x^4 + 4x^3 - 3x^2 - 2x + 1 = 0$.
12. $3x^4 + 13x^3 + 30x - 2 = 0$.

132. Finding the rational roots. A rational number is a real number that can be expressed as the quotient of two integers. Thus, $\frac{4}{7}$, $-\frac{1}{2}$, 0, -7, and 6 are rational numbers, whereas $\sqrt{5}$, $-\sqrt[3]{2}$, and $2 + \sqrt{7}$ are irrational numbers.

The rational roots of an equation may be found by a process of testing various possibilities which are obtained by use of the following

Theorem on rational roots (theorem 7). *If a rational number $\frac{c}{d}$, a fraction in its lowest terms, is a root of the equation*

$$a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

with integral coefficients, then c is a factor of the constant term a_n, and d is a factor of a_0.

Proof. Since $\frac{c}{d}$ is a root,

$$a_0\left(\frac{c}{d}\right)^n + a_1\left(\frac{c}{d}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{c}{d}\right) + a_n = 0.$$

Multiply by d^n to clear of fractions:

$$a_0c^n + a_1c^{n-1}d + \cdots + a_{n-1}cd^{n-1} + a_nd^n = 0. \tag{1}$$

Transpose the last term and then factor the left side:

$$c(a_0c^{n-1} + a_1c^{n-2}d + \cdots + a_{n-1}d^{n-1}) = -a_nd^n. \tag{2}$$

Since c, d, and the a's are integers, the left side of (2) is an integer having c as a factor. Hence the right side, $-a_nd^n$, must have c as a factor. Since $\frac{c}{d}$ is a fraction in its lowest terms, c and d have no common factors (except 1 and -1). Consequently c and d^n have no common factors. Since c cannot be a factor of d^n, it must be a factor of a_n.

Returning to (1) and transposing the first term, we get

$$a_1c^{n-1}d + \cdots + a_{n-1}cd^{n-1} + a_nd^n = -a_0c^n.$$

Obviously d is a factor of the left side and hence must be a factor of $-a_0c^n$. Since d and c^n have no common factors, d must be a factor of a_0.

Corollary. If the equation

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

has integral coefficients, then any rational root is an integer and is a factor of a_n.

Proof. By theorem 7, if $\frac{c}{d}$ is a rational root, then d must be a factor of 1. Hence $\frac{c}{d}$ is an integer which is a factor of a_n.

Example 1. Find the rational roots of

$$3x^3 + 19x^2 - 9x - 5 = 0.$$

Solution.

(1) Apply Descartes' rule.

Number of positive roots: 1.
Number of negative roots: 2 or 0.

(2) Apply theorem on rational roots.

Possible numerators: 1, 5.
Possible denominators: 1, 3.
Possible rational roots: $\pm(1, \frac{1}{3}, 5, \frac{5}{3})$.

Arrange in order of increasing numerical size. (This enables us to make full use of the theorem on the limits for the roots).

$$\tfrac{1}{3}, 1, \tfrac{5}{3}, 5 \quad \text{and} \quad -\tfrac{1}{3}, -1, -\tfrac{5}{3}, -5.$$

(3) Test positive * possibilities $\frac{1}{3}$, 1, etc.

$$\begin{array}{l|l} 3 + 19 - 9 - 5 & \tfrac{1}{3} \\ \quad + 1 & \\ \hline 3 + 20 & \end{array} \qquad \begin{array}{rrrr|l} 3 + & 19 - & 9 - & 5 & 1 \\ + & 3 + & 22 + & 13 & \\ \hline 3 + & 22 + & 13 + & 8 & \end{array}$$

The synthetic division for $\frac{1}{3}$ need not be continued because the next number and all succeeding ones will be fractions. The remainder cannot possibly be zero.

In the synthetic division for 1, we find that all numbers in the third line are positive. Hence 1 is an upper limit for the roots. We need not test $\frac{5}{3}$ and 5.

(4) Test negative possibilities $-\frac{1}{3}$, -1, etc.

$$\begin{array}{rrrr|l} 3 + & 19 - & 9 - & 5 & -\tfrac{1}{3} \\ - & 1 - & 6 + & 5 & \\ \hline 3 + & 18 - & 15 + & 0 & \end{array} \qquad \text{Hence } -\tfrac{1}{3} \text{ is a root.}$$

We now see that the original equation can be written in the form

$$(x + \tfrac{1}{3})(3x^2 + 18x - 15) = 0.$$

Discarding the factor $(x + \frac{1}{3})$ which corresponds to the root $-\frac{1}{3}$, we have the **depressed equation**

$$3x^2 + 18x - 15 = 0.$$

Divide by 3: $$x^2 + 6x - 5 = 0.$$

* In this case it would be equally logical to begin with the negative possibilities.

Solving this quadratic equation by completing the square, we get $x = -3 \pm \sqrt{14}$.

The roots of the given equation are

$$r_1 = -\tfrac{1}{3}, \quad r_2 = -3 + \sqrt{14} = .742, \quad r_3 = -3 - \sqrt{14} = -6.742.$$

Notice that we have one positive root and two negative roots. This agrees with Descartes' rule, which applies to *all real roots*, rational and irrational.

Example 2. Find the rational roots of

$$9x^5 + 19x^4 + 65x^3 + 115x^2 + 66x + 6 = 0.$$

Solution.

Number of positive roots: 0.
Number of negative roots: 5, 3, or 1.
Possible numerators (factors of 6): 1, 2, 3, 6.
Possible denominators (factors of 9): 1, 3, 9.
Possible rational roots: $-(1, 2, 3, 6, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{6}{3}, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{6}{9})$.
Cast out duplicates and arrange in order of size:

$$-\tfrac{1}{9}, \quad -\tfrac{2}{9}, \quad -\tfrac{1}{3}, \quad -\tfrac{2}{3}, \quad -1, \quad -2, \quad -3, \quad -6.$$

$$\begin{array}{l|l} 9 + 19 + 65 + 115 + 66 + 6 & -\frac{1}{9} \\ \quad - \ 1 - \ 2 - \ \ 7 - 12 - 6 & \\ \hline 9 + 18 + 63 + 108 + 54 + 0 & \end{array} \qquad r_1 = -\tfrac{1}{9}$$

The depressed equation $9x^4 + 18x^3 + 63x^2 + 108x + 54 = 0$ should be divided by 9:

$$x^4 + 2x^3 + 7x^2 + 12x + 6 = 0.$$

Possible rational roots: $-1, -2, -3, -6$.

$$\begin{array}{l|l} 1 + 2 + 7 + 12 + 6 & -1 \\ \quad - 1 - 1 - \ 6 - 6 & \\ \hline 1 + 1 + 6 + \ 6 + 0 & \end{array} \qquad r_2 = -1$$

For the new depressed equation $x^3 + x^2 + 6x + 6 = 0$, the possible rational roots are $-1, -2, -3, -6$. Although -1 was found to be a root of the first depressed equation, we should test it again to see if it is a double root.

$$\begin{array}{l|l} 1 + 1 + 6 + 6 & -1 \\ \quad - 1 + 0 - 6 & \\ \hline 1 + 0 + 6 + 0 & \end{array} \qquad r_3 = -1$$

The roots of the final depressed equation $x^2 + 6 = 0$ are

$$r_4 = i\sqrt{6}, \qquad r_5 = -i\sqrt{6}.$$

Notice that the various synthetic divisions have reduced the original equation to the factored form

$$(x + \tfrac{1}{9})9(x + 1)^2(x^2 + 6) = 0.$$

Exercise 69

Find all the rational roots of the following equations. When possible, find all the roots.

1. $x^3 - 12x^2 + 24x - 8 = 0.$

2. $x^3 + 7x^2 + 16x + 12 = 0.$

3. $x^3 - 7x - 6 = 0.$

4. $x^3 + x^2 - 2 = 0.$

5. $15x^3 + 13x^2 - 3x - 1 = 0.$

6. $10x^3 - 42x^2 + 3x + 1 = 0.$

7. $18x^3 + 51x^2 + 14x + 1 = 0.$

8. $12x^3 - 23x^2 + 9x - 1 = 0.$

9. $5x^3 + 4x^2 + 8x + 7 = 0.$

10. $7x^3 + 5x = 11.$

11. $3x^3 + 10x = 8x^2 + 4.$

12. $3x^3 + 19x^2 + 12x + 2 = 0.$

13. $x^4 - 19x^2 + 30x = 0.$

14. $x^3 - 4x^2 - 20x + 48 = 0.$

15. $8x^3 + 22x^2 - 7x - 3 = 0.$

16. $x^3 - \frac{3}{2}x^2 - 4x + 6 = 0.$

17. $2x^3 + 11x^2 + 10x - 14 = 0.$

18. $9x^3 - 41x^2 + 11x + 5 = 0.$

19. $x^4 + 4x^3 + 6x^2 + 4x + 1 = 0.$

20. $4x^4 - 28x^3 + 57x^2 - 32x + 5 = 0.$

21. $9x^4 - 24x^3 + 31x^2 - 14x + 2 = 0.$

22. $6x^4 + 5x^3 + 31x^2 + 25x + 5 = 0.$

23. $x^3 + 8x^2 + 10x - 24 = 0.$

24. $2x^4 + 5x^3 + 3x^2 + 15x - 9 = 0.$

25. $7x^4 + 6x^3 - 22x^2 - 18x + 3 = 0.$

26. $3x^4 + 34x^3 + 74x^2 - 6x - 9 = 0.$

27. $5x^5 - 11x^4 + 17x^3 - 23x^2 + 14x - 2 = 0.$

28. $x^5 + 8x^4 + 22x^3 + 26x^2 + 13x + 2 = 0.$

29. $4x^4 - 11x^2 + 3x + 4 = 0.$

30. $8x^4 - 4x^3 - 26x^2 + 25x - 6 = 0.$

133. Irrational roots. The irrational roots of an equation may be found by any one of several methods, two of which are described in Art. 134 and 136. The following theorem is the basis of both methods.

Location theorem. *If $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one real root between a and b.*

Proof. If we graph $f(x)$ as a function of x, the two points that correspond to $x = a$ and $x = b$ will lie on opposite sides of the x-axis. Since $f(x)$ represents a polynomial, its graph will be a continuous curve that must cross the x-axis at least once between $x = a$ and $x = b$. To this point of intersection there corresponds a real root of $f(x) = 0$.

134. Finding the irrational roots by successive graphs.

Example 1. Find to three decimal places the real root of

$$2x^3 + 6x - 5 = 0.$$

Solution. By Descartes' rule, there is one positive root and no negative root. The possible rational roots are $\frac{1}{2}$, 1, $\frac{5}{2}$, 5. Synthetic division shows that $\frac{1}{2}$ and 1 are not roots and that 1 is an upper limit.

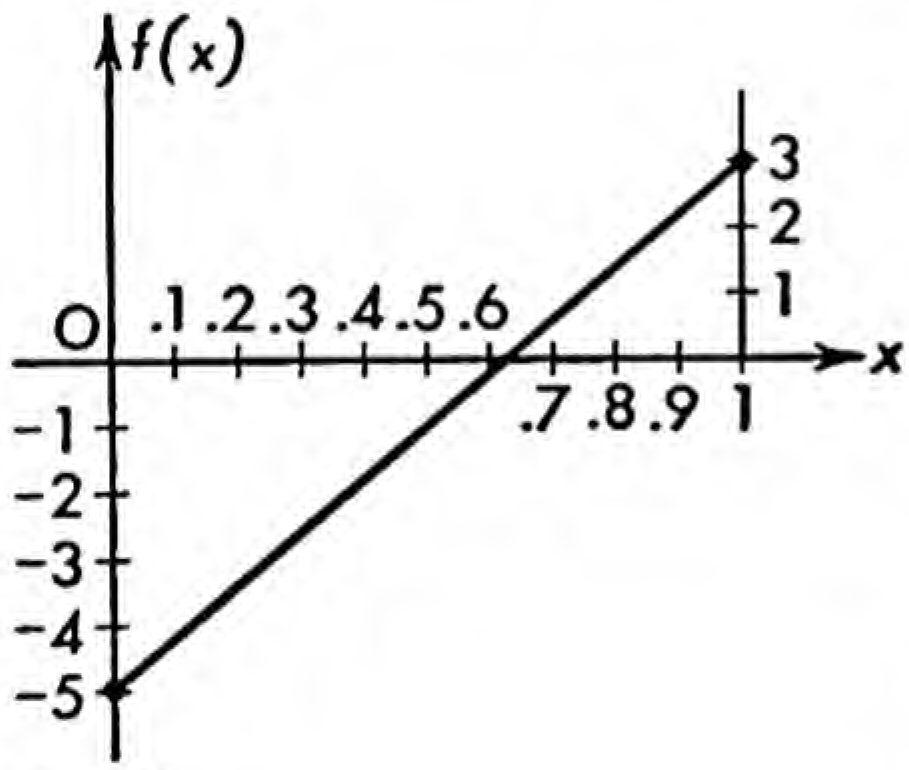


FIG. 31

Since the only real root is positive, we shall compute the value of $f(x) = 2x^3 + 6x - 5$ for $x = 0$, 1, etc. We find $f(0) = -5$ and $f(1) = 3$. Hence, by the location theorem, the root is between 0 and 1.

The graph of $f(x)$ between $x = 0$ and $x = 1$ can be approximated by a straight line (Fig. 31) connecting the points $(0, -5)$ and $(1, 3)$. This line seems to cross the x-axis at $.6^+$. Accordingly we find $f(.6)$ by synthetic division.

$$\begin{array}{rrrr|l} 2 + 0 & +6 & -5 & & \underline{.6} \\ +1.2 + & .72 & +4.032 & & \\ \hline 2 + 1.2 + & 6.72 & - .968 & & \end{array}$$

Since $f(.6)$ is negative, the root is larger than .6.

Next we find $f(.7) = -.114$. Hence the root is larger than .7. Then we find $f(.8) = +.824$. The fact that $f(.7)$ and $f(.8)$ have opposite signs means that the root is between .7 and .8. Moreover, since

$-.114$ is closer to zero than is $+.824$, we conclude that the root is probably * closer to .7. Hence, to one decimal accuracy

$$r = .7^{+}.$$

To obtain the root to two decimal places, we enlarge that portion of the curve that lies between .7 and .8, and approximate it by using the straight line (Fig. 32) connecting the points $(.7, -.114)$ and $(.8, +.824)$. This line seems to cross the x-axis at $.71^{+}$. Accordingly, we next compute $f(.71)$. This can be done by synthetic division or by substitution, using a table of cubes (Table I). Employing the latter method, we find

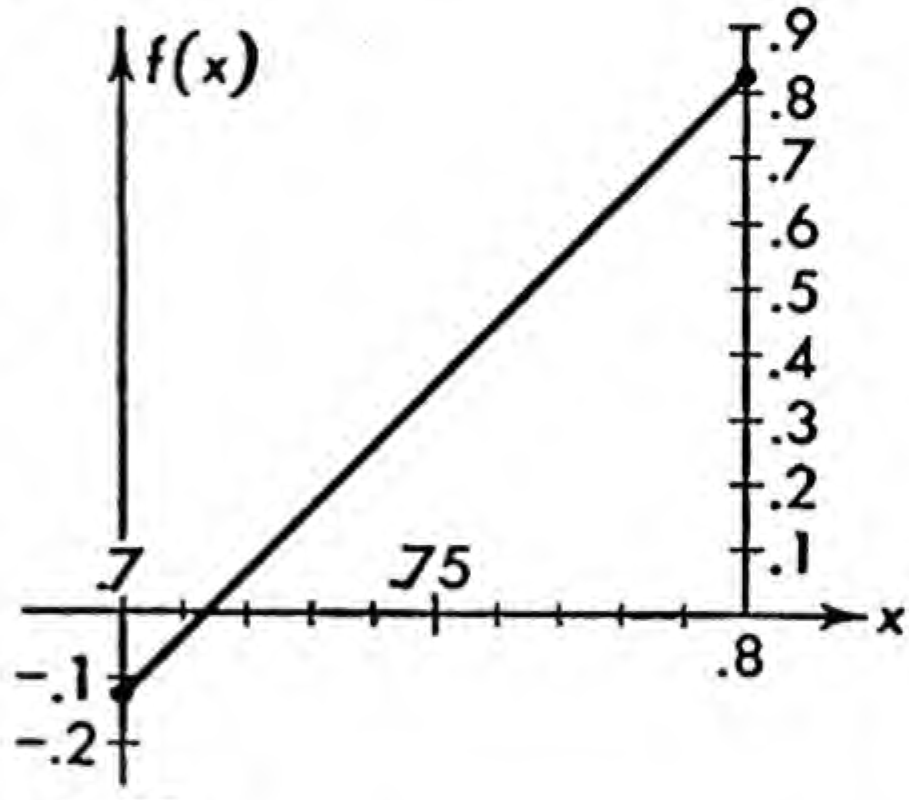


FIG. 32

$$\begin{aligned} f(.71) &= 2(.71)^3 + 6(.71) - 5 \\ &= 2(.357911) + 4.26 - 5 \\ f(.71) &= -.024178. \end{aligned}$$

Similarly, $\quad f(.72) = +.066496.$

Hence, to two-decimal accuracy,

$$r = .71^{+}.$$

To obtain the root to three decimal places, we approximate the curve between .71 and .72 by using the straight line (Fig. 33) connecting the points $(.71, -.024178)$ and $(.72, .066496)$. This line seems to cross the x-axis at $.713^{-}$. Hence, to three-decimal accuracy,

$$r = .713.$$

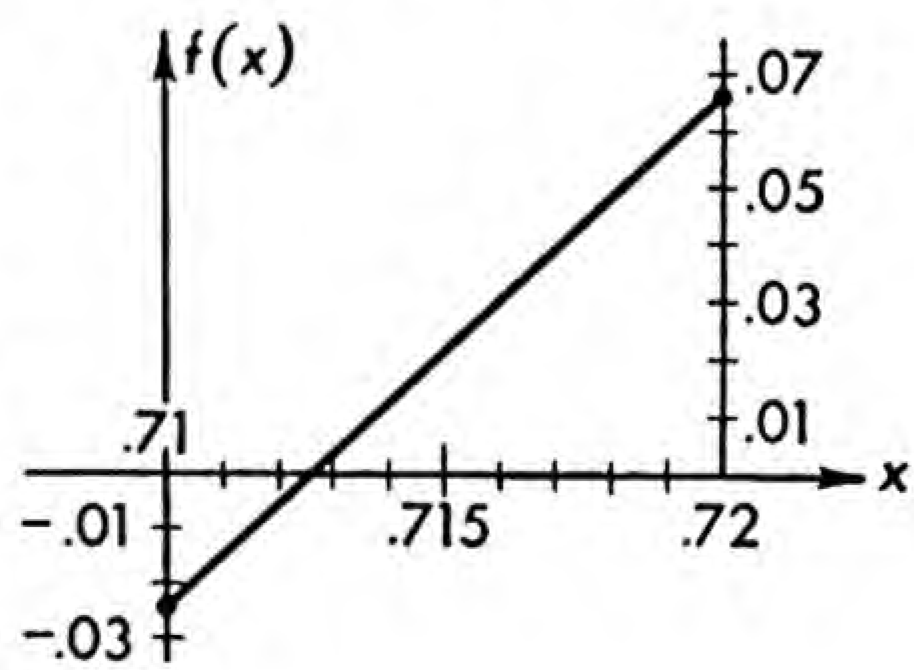


FIG. 33

It is advisable to retain all computed values of $f(x)$ and place

* This could be definitely established by showing that $f(.75)$ is positive.

them in a table, which, as it is being filled in, will show the progress made in isolating the root. For this example we have the following table.

x	0	.6	.7	.71		.72	.8	1
$f(x)$	-5	$-.968$	$-.114$	$-.0242$		.0665	.824	3

In finding a negative irrational root of $f(x) = 0$ it is convenient to find the corresponding positive root of $f(-x) = 0$.

Exercise 70

Find to two decimal places the real roots of the following equations.

1. $x^3 + 7x - 1 = 0$. **2.** $x^3 + 9x - 1 = 0$.

3. $x^3 + x^2 - 13 = 0$. **4.** $x^3 - x^2 - 11 = 0$.

5. $x^4 - 21x + 8 = 0$. **6.** $x^3 + x^2 - 5x + 2 = 0$.

Find to three decimal places the real roots of the following equations.

7. $x^3 + 5x - 1 = 0$. **8.** $x^3 + 2x - 1 = 0$.

9. $7x^3 + 4x - 1 = 0$. **10.** $x^3 + 4x - 7 = 0$.

11. $x^3 + x^2 + 10x - 17 = 0$. **12.** $x^3 + x - 7 = 0$.

13. Find to two decimal places the *two* real roots between 0 and 1: $2x^3 + 2x^2 - 4x + 1 = 0$.

Find exactly or correct to one decimal place the real roots of the following equations.

14. $x^4 - x^3 - 4x^2 + 5x - 1 = 0$.

Hint. Use the depressed equation after the rational roots have been found.

15. $x^4 - 2x^3 + x^2 - 3x + 2 = 0$. **16.** $x^5 + 2x^4 - 3x^3 - x^2 - 3x = 0$.

135. Transformation of an equation to decrease its roots. The preceding article gives a general method of finding the irrational roots of any equation. In Art. 136 we shall discuss other means of finding the irrational roots of an integral rational equation. This method involves a process of finding a new equation whose roots are less, by a given amount, than the roots of the original equation. We can accomplish this by using the following

Rule. To decrease the roots of $f(x) = 0$ by an amount h:

1. Divide $f(x)$ by $(x - h)$; let R_n be the remainder.
2. Divide the quotient by $(x - h)$; let R_{n-1} be the remainder.
3. Continue this process to n divisions. (The polynomial $f(x)$ is of degree n.)
4. The last quotient a_0 and the remainders $R_1, R_2, \cdots, R_{n-1}, R_n$ are the coefficients of the new equation:

$$a_0y^n + R_1y^{n-1} + R_2y^{n-2} + \cdots + R_{n-1}y + R_n = 0.$$

Example 1. Transform $x^4 - 9x^3 + 3x^2 + 89x - 84 = 0$ into an equation whose roots are less by 2.

Solution.

$$\begin{array}{rrrrrl}
1 & -9 & +3 & +89 & -84 & \underline{|2} \\
 & +2 & -14 & -22 & +134 & \\ \hline
1 & -7 & -11 & +67 & \multicolumn{1}{|r}{+50} & \qquad R_4 = 50 \\
 & +2 & -10 & -42 & & \\ \hline
1 & -5 & -21 & \multicolumn{1}{|r}{+25} & & \qquad R_3 = 25 \\
 & +2 & -6 & & & \\ \hline
1 & -3 & \multicolumn{1}{|r}{-27} & & & \qquad R_2 = -27 \\
 & +2 & & & & \\ \hline
1 & -1 & & & & \qquad R_1 = -1
\end{array}$$

The new equation is

$$y^4 - y^3 - 27y^2 + 25y + 50 = 0.$$

The student should check the result by showing that the roots of the given equation are -3, 1, 4, 7, whereas the roots of the new equation are -5, -1, 2, 5.

Proof of the rule. Let

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0 \tag{1}$$

be the original equation. Replace x with $(y + h)$:

$$f(y+h) = a_0(y+h)^n + a_1(y+h)^{n-1} + \cdots + a_{n-1}(y+h) + a_n = 0, \tag{2}$$

which, on expanding and collecting terms, takes the form

$$f(y + h) = a_0y^n + A_1y^{n-1} + \cdots + A_{n-1}y + A_n = 0. \tag{3}$$

Since $x = y + h$, or $y = x - h$, it follows that each root $x = r$ of equation (1) will correspond to a root $y = r - h$ of equation (3). This means that the roots of (3) are the roots of (1) decreased by h.

The A's can be determined by changing back to (1) again by setting $y = x - h$ in (3):

$$f(x) = a_0(x-h)^n + A_1(x-h)^{n-1} + \cdots + A_{n-1}(x-h) + A_n = 0. \quad (4)$$

This is the original equation $f(x) = 0$ arranged in powers of $(x - h)$. From the form of (4), we see that A_n is the remainder R_n when $f(x)$ is divided by $(x - h)$; A_{n-1} is the remainder R_{n-1} when the previously obtained quotient is divided by $(x - h)$; etc.

Exercise 71

Transform each equation to decrease the roots as indicated.

1. $x^3 - 12x^2 + 41x - 30 = 0$; decrease roots by 3.

2. $x^3 - 10x^2 + 11x + 70 = 0$; decrease roots by 1.

3. $5x^3 + 6x^2 - 7 = 0$; decrease roots by .2.

4. $3x^3 - 17x - 11 = 0$; decrease roots by 2.

5. $x^4 - 4x^3 + 5x^2 + 6x - 7 = 0$; decrease roots by 1.

6. $x^4 - 7x^2 + 8x + 1 = 0$; *increase* roots by 3.

7. $2x^4 - 9x^3 + 8x + 7 = 0$; decrease roots by 4.

8. $x^3 + .3x^2 - 1.6x + 1.28 = 0$; decrease roots by .7.

9. (*a*) Decrease the roots of $x^3 + 10x^2 + 30x - 56 = 0$ by 1.
(*b*) Decrease the roots of $w^3 + 13w^2 + 53w - 15 = 0$ by .2.
(*c*) Decrease the roots of $t^3 + 13.6t^2 + 58.32t - 3.872 = 0$ by .06.

136. Finding the irrational roots by Horner's method.

Example 1. Find the real roots of

$$f(x) = x^3 + 2x^2 - 4x - 17 = 0. \quad (1)$$

Solution. By Descartes' rule, there is one positive root and two or no negative roots. A graph (Fig. 34) of $f(x)$ shows that there is no negative root and that the positive root is between 2 and 3.

We shall transform the equation to decrease its roots by 2.

$$
\begin{array}{l}
1 + 2 - \ 4 - 17 \ \underline{|2} \\
\underline{\ \ + 2 + \ 8 + \ 8} \\
1 + 4 + \ 4 \,| - \ 9 \\
\underline{\ \ + 2 + 12 \,|} \\
1 + 6 \,| + 16 \\
\underline{\ \ + 2 \,|} \\
1 + 8
\end{array}
$$

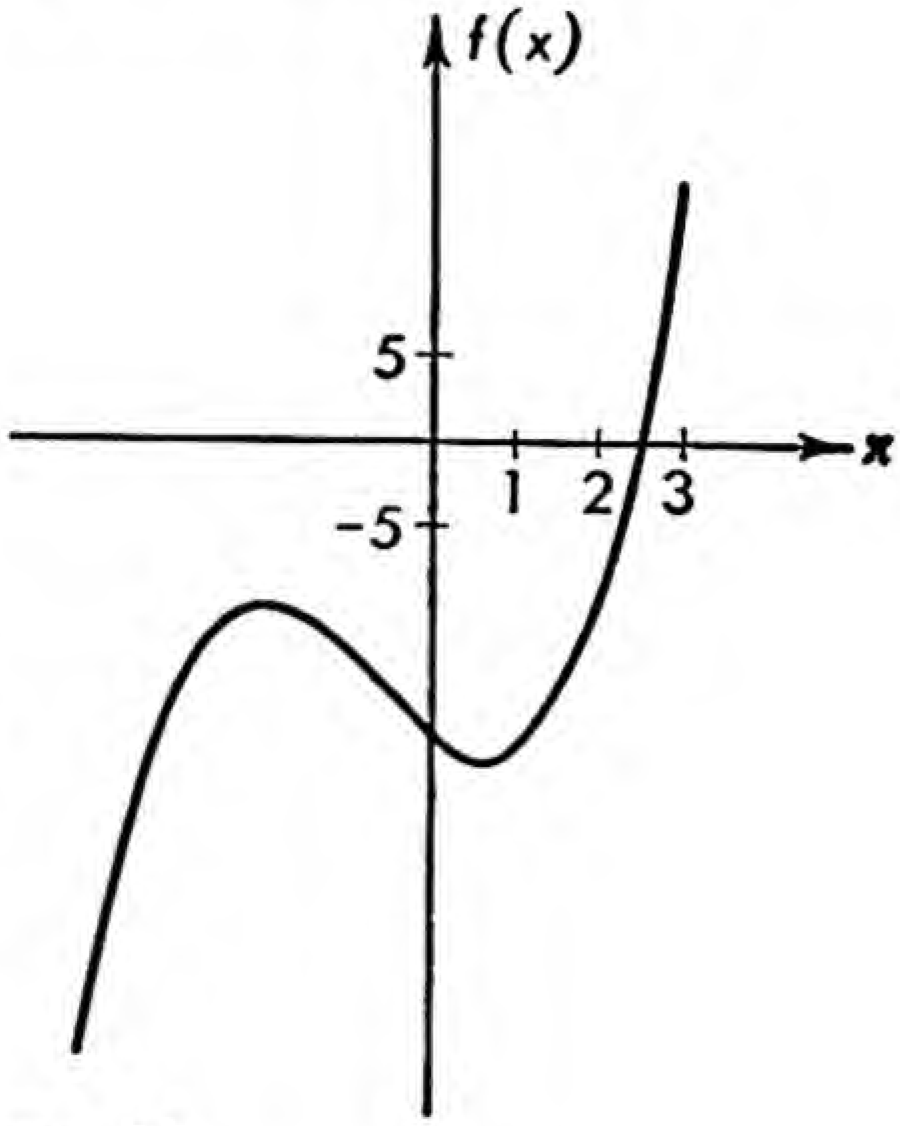


FIG. 34

The new equation,

$$f_1(x_1) = x_1^3 + 8x_1^2 + 16x_1 - 9 = 0, \quad (2)$$

has a root between 0 and 1. Since x is small, x^3 is very small and we can obtain a good approximation of the root by solving the quadratic equation $8x_1^2 + 16x_1 - 9 = 0$. We find the positive root to be $-1 + \sqrt{\frac{17}{8}} = .4^+$. This means that the root of (2) is *probably* between .4 and .5. Using synthetic division we find $f(.4) = -1.256$, and $f(.5) = +1.125$. By the location theorem, we are now certain that the root of (2) is between .4 and .5.

Transform equation (2) to decrease its roots by .4.

$$
\begin{array}{l}
1 + 8 \ \ \ + 16 \ \ \ \ - 9 \ \ \ \underline{|.4} \\
\underline{\ \ + \ .4 + \ 3.36 + 7.744} \\
1 + 8.4 + 19.36 \,| - 1.256 \\
\underline{\ \ + \ .4 + \ 3.52 \,|} \\
1 + 8.8 \,| + 22.88 \\
\underline{\ \ + \ .4 \,|} \\
1 + 9.2
\end{array}
$$

The new equation,

$$f_2(x_2) = x_2^3 + 9.2x_2^2 + 22.88x_2 - 1.256 = 0, \qquad (3)$$

has a root between 0 and .1. We can approximate this root by solving the linear equation $22.88x_2 - 1.256 \doteq 0$, and obtaining $x_2 = .05^+$. Computing $f(.05) = -.088875$ and $f(.06) = +.150136$, we see that the root of (3) lies between .05 and .06.

Transform equation (3) to decrease its roots by .05.

$$
\begin{array}{llll|l}
1 & +\,9.2 & +\,22.88 & -\,1.256 & .05 \\
 & +\ \ .05 & +\ \ .4625 & +\,1.167125 & \\ \hline
1 & +\,9.25 & +\,23.3425 & -\ \ .088875 & \\
 & +\ \ .05 & +\ \ .465 & & \\ \hline
1 & +\,9.3 & +\,23.8075 & & \\
 & +\ \ .05 & & & \\ \hline
1 & +\,9.35 & & &
\end{array}
$$

The new equation,

$$f_3(x_3) = x_3^3 + 9.35x_3^2 + 23.8075x_3 - .088875 = 0, \qquad (4)^*$$

has a root between 0 and .01. Approximate this root by solving the equation $23.8075x_3 - .088875 = 0$, and finding $x_3 = .0037^+$.

The real root of the given equation $f(x) = 0$ is

$$x = 2 + .4 + .05 + .0037 = 2.4537.$$

The number in the fourth decimal place may be incorrect but we are fairly certain that the root to three decimal places is 2.454.

The solution may be arranged in the following compact form.

$$
\begin{array}{llll|l|l}
1 & +\,2 & -\ \ 4 & -\,17 & 2 & \\
 & +\,2 & +\ \ 8 & +\ \ 8 & & \\ \hline
1 & +\,4 & +\ \ 4 & -\ \ 9 & & \\
 & +\,2 & +\,12 & & & \\ \hline
1 & +\,6 & +\,16 & & & \\
 & +\,2 & & & & \\ \hline
1 & +\,8 & +\,16 & -\ \ 9 & .4 & 8x_1^2 + 16x_1 - 9 = 0 \\
 & +\ \ .4 & +\ \ 3.36 & +\ \ 7.744 & & x_1 = .4. \\ \hline
1 & +\,8.4 & +\,19.36 & -\ \ 1.256 & & \\
 & +\ \ .4 & +\ \ 3.52 & & & \\ \hline
1 & +\,8.8 & +\,22.88 & & & \\
 & +\ \ .4 & & & & \\ \hline
1 & +\,9.2 & +\,22.88 & -\ \ 1.256 & .05 & 22.88x_2 - 1.256 = 0 \\
 & +\ \ .05 & +\ \ .4625 & +\ \ 1.167125 & & x_2 = .05. \\ \hline
1 & +\,9.25 & +\,23.3425 & -\ \ .088875 & & \\
 & +\ \ .05 & +\ \ .4650 & & & \\ \hline
1 & +\,9.30 & +\,23.8075 & & & 23.8075x_3 - .088875 = 0 \\
 & +\ \ .05 & & & & x_3 = .0037. \\ \hline
1 & +\,9.35 & & & & x = 2.4537.
\end{array}
$$

* The three roots of (4) are those of the original equation 1, each decreased

Horner's method for finding a positive irrational root.

1. *Locate the root between successive integers* a *and* (a + 1).
2. *Transform the equation to decrease its roots by* a.
3. *Locate the root between successive tenths* b *and* (b + .1).
4. *Transform the equation to decrease its roots by* b.
5. *Locate the root between successive hundredths, etc.*
6. *Continue this process to find the root to the desired accuracy.*

To find a *negative* root of $f(x) = 0$ by Horner's method, find the corresponding *positive* root of $f(-x) = 0$, and multiply by -1.

137. Suggested procedure for finding the real roots of an equation.

1. *Apply Descartes' rule of signs.*

2. *Find all rational roots. After a rational root has been found, remove the corresponding factor and use the depressed equation.*

3. *Let* $f(x) = 0$ *represent the depressed equation after all rational roots have been removed. Sketch a graph of* $f(x)$*, assigning integral values to* x *until upper and lower limits have been reached. This graph locates the real roots between integers.*

4. *Find the irrational roots of* $f(x) = 0$ *by the method of successive graphs or by Horner's method.*

Exercise 72

*Find the real root of each equation to three * decimal places.*

1. $x^3 - 6x^2 + 14x - 13 = 0.$ **2.** $x^3 + 3x^2 - 3x - 17 = 0.$

3. $x^3 + 2x^2 + x - 11 = 0.$ **4.** $x^3 - 9x^2 + 36x - 55 = 0.$

5. $x^3 + 4x^2 + 5x - 11 = 0.$ **6.** $x^3 - 2x^2 - 9x - 7 = 0.$

*Find all the real roots of each equation to three * decimal places.*

7. $x^3 - 8x^2 + 16x - 1 = 0.$

8. $2x^3 - 4x^2 - 2x + 5 = 0.$

9. $x^4 - 6x^3 - x^2 - 2x + 5 = 0.$

10. $x^4 - 9x^3 + 22x^2 - 21x + 13 = 0.$

by 2.45. The other two roots of (1) can be approximated by solving (4) with the constant term replaced by zero and then increasing the roots by 2.45.

* Two decimal places if so directed by the instructor.

11. $x^4 - 2x^3 - 8x^2 - 13x - 14 = 0.$

12. $x^4 - 6x^3 + 8x^2 + 2x - 1 = 0.$

13. $2x^4 - 17x^3 + 45x^2 - 41x + 7 = 0.$

14. $7x^4 - 7x^3 + 4x^2 - 5x + 1 = 0.$

15. $x^4 - 2x^3 + 5x^2 - 11x + 2 = 0.$

16. $2x^5 + 5x^3 - x^2 = 0.$

17. Approximate $\sqrt[3]{5}$ by finding to three decimal places the real root of $x^3 = 5$.

18. Approximate $\sqrt[4]{57}$ by finding to three decimal places the positive root of $x^4 = 57$.

19. When floating in water, a sphere of radius r and specific gravity s will sink to a depth x, where x is the smaller positive root of $x^3 - 3rx^2 + 4r^3s = 0$. If the specific gravity of a cork sphere is .2, how deep will it sink if its radius is 1 inch? (Obtain solution to three decimal places.)

20. The dimensions of a rectangular box are 2, 3, and 5 inches. The volume is to be doubled by increasing each edge by the same amount. Find to three decimal places the amount to be added.

138. Coefficients expressed in terms of the roots. Let r_1, r_2, r_3 be the roots of the equation

$$x^3 + b_1x^2 + b_2x + b_3 = 0. \tag{1}$$

By Art. 127, we can write the equation in the form

$$(x - r_1)(x - r_2)(x - r_3) = 0, \tag{2}$$

which, multiplied out, becomes

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0. \tag{3}$$

Using the corollary of Art. 125, to compare (1) and (3), we see that

$$\begin{aligned} b_1 &= -(r_1 + r_2 + r_3) \\ &= -(\text{the sum of the roots}), \\ b_2 &= +(r_1r_2 + r_1r_3 + r_2r_3) \\ &= +(\text{the sum of the products of the roots, taken two at a time}), \\ b_3 &= -r_1r_2r_3 \\ &= -(\text{the product of the roots}). \end{aligned}$$

In a similar manner we can establish the following general

Theorem. *If* $r_1, r_2, \cdots, r_n$ *are the roots of*

$$x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n = 0, \tag{4}$$

then

$b_1 = -$(*the sum of the roots*),
$b_2 = +$(*the sum of the products of the roots, taken two at a time*),
$b_3 = -$(*the sum of the products of the roots, taken three at a time*),
. . . .
$b_n = (-1)^n$(*the product of the roots*).

If the given equation is of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_n = 0,$$

divide through by a_0 and reduce it to the form (4).

Exercise 73

Use the theorem of Art. 138 *to find an equation having the following roots.*

1. r_1, r_2, r_3, r_4.
2. $\frac{1}{2}, 2, 3, 4$.
3. $0, 1, 2, -3$.
4. $5, 3i, -3i$.
5. $-\frac{1}{3}, 2, 6$.
6. $1, 1, 2, 3, 5$.

In each of the following equations, state (a) *the sum of the roots,* (b) *the sum of the products of the roots, taken two at a time,* (c) *the product of the roots.*

7. $4x^3 + 5x^2 - 6x - 7 = 0$.

8. $7x^3 - 8x^2 - 9 = 0$.

9. The equation $x^3 - 10x^2 + px + q = 0$ has the roots 3 and 5. Find the other root and the values of p and q.

10. The equation $5x^3 + px^2 + qx - 24 = 0$ has the roots 2 and 4. Find the other root and the values of p and q.

11. In the equation $2x^3 - 7x^2 - 8x + q = 0$ one root is the negative of another. Find the roots and the value of q.

12. The roots of the equation $x^3 - 9x^2 + 2x + q = 0$ can be arranged to form an arithmetic progression. Find the roots and the value of q.

13. The roots of the equation $x^3 + kx^2 - 54x + 216 = 0$ can be arranged to form a geometric progression. Find the roots and the value of k.

14. The equation $x^4 - 16x^3 + 94x^2 + px + q = 0$ has two double roots. Find the roots and the values of p and q.

chapter 17

Logarithms, exponential equations

139. The uses of logarithms. Logarithms are used to shorten the labor involved in computing products and quotients, raising to a power, and extracting roots. For example, the operation of *multiplying* 23.45 by 678.9 may be reduced to the operation of *adding* the logarithms of these numbers, namely 1.3702 and 2.8318.

140. Definition of a logarithm. *The* **logarithm** *of a number to a given base is the exponent which must be placed on the base to produce the number.* Thus the logarithm of 8 to the base 2 (written $\log_2 8$) is 3 because $2^3 = 8$.

Illustrations.

$$
\begin{array}{lll}
\log_3 81 = 4 & \text{because} & 3^4 = 81. \\
\log_6 1 = 0 & \text{because} & 6^0 = 1. \\
\log_7 \frac{1}{7} = -1 & \text{because} & 7^{-1} = \frac{1}{7}. \\
\log_2 \frac{1}{8} = -3 & \text{because} & 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}. \\
\log_{49} 7 = \frac{1}{2} & \text{because} & 49^{\frac{1}{2}} = \sqrt{49} = 7. \\
\log_8 4 = \frac{2}{3} & \text{because} & 8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4. \\
\log_5 5 = 1 & \text{because} & 5^1 = 5.
\end{array}
$$

The definition of a logarithm implies that

if $$\log_b N = x,$$
then $$b^x = N.$$

These two equations, the former logarithmic and the latter exponential, say the same thing in two different ways. We shall assume in

further discussions that N is a positive number and that b is a positive number different from 1.

The following facts are proved in higher mathematics. We shall merely assume their truth.

I. Irrational exponents have meaning and obey the five laws of exponents (Art. 46), provided the base is positive. For example, $\log_{10} 2$ is an irrational number which is approximately 0.3010. If the exponent $\frac{3{,}010}{10{,}000}$ is placed on 10, the result is 2 *approximately*.

II. If N is a positive number, then there is one and only one number x that satisfies the equation $\log_b N = x$.

Exercise 74

Find the value of each of the following logarithms.

1. $\log_{10} 100$. **2.** $\log_{10} 1000$.
3. $\log_7 7$. **4.** $\log_{10} 1$.
5. $\log_{10} .1$. **6.** $\log_{10} .01$.
7. $\log_3 1$. **8.** $\log_{\frac{1}{2}} 2$.
9. $\log_7 \frac{1}{49}$. **10.** $\log_{25} 5$.
11. $\log_{64} 4$. **12.** $\log_{27} \frac{1}{9}$.
13. $\log_4 8$. **14.** $\log_{\sqrt{6}} 36$.
15. $\log_{\frac{1}{8}} 2$. **16.** $\log_{a^4} a^6$.

Find the unknown N, b, or x in each of the following.

17. $\log_2 N = 7$. **18.** $\log_5 N = -3$.
19. $\log_b 9 = -1$. **20.** $\log_b 16 = 4$.
21. $\log_b \frac{1}{8} = -\frac{3}{2}$. **22.** $\log_b 32 = \frac{5}{2}$.
23. $\log_{32} N = \frac{4}{5}$. **24.** $\log_{27} N = \frac{4}{3}$.
25. $\log_9 N = -2$. **26.** $\log_b \frac{1}{2} = -\frac{1}{4}$.
27. $\log_9 \sqrt{3} = x$. **28.** $\log_3 3^{47} = x$.

Express as a logarithmic equation.

29. $3^5 = 243$. **30.** $8^{\frac{7}{3}} = 128$.
31. $b^r = s$. **32.** $4^0 = 1$.

Express as an exponential equation.

33. $\log_2 512 = 9$. **34.** $\log_a w = z$.
35. $\log_{100} .001 = -\frac{3}{2}$. **36.** $\log_{\frac{1}{3}} 81 = -4$.

Identify as true or false and give reasons.

37. $\log_b b^r = r.$

38. $\log_{\frac{1}{8}} \sqrt{2} = -\frac{1}{6}.$

39. $\log_2 8^m = 3m.$

40. $a^{\log_a x} = x.$

141. Properties of logarithms. As a consequence of the definition of a logarithm, we have three properties or laws of logarithms. They are used in computations involving logarithms.

Property 1. The logarithm of a product is equal to the sum of the logarithms of the factors; i.e.,

$$\log MN = \log M + \log N.^*$$

Proof. Let $x = \log_b M$ and $y = \log_b N$.

Express in exponential form: $M = b^x$ and $N = b^y$.
Multiply the equations: $MN = b^x b^y = b^{x+y}$.
Change to logarithmic form: $\log_b MN = x + y$.
$\log_b MN = \log_b M + \log_b N$.

The proof is similar for a product of three or more factors.

Illustrations.

$$\log 15 = \log 3 \cdot 5 = \log 3 + \log 5.$$
$$\log 70 = \log 2 \cdot 5 \cdot 7 = \log 2 + \log 5 + \log 7.$$

Property 2. The logarithm of a fraction is equal to the logarithm of the numerator minus the logarithm of the denominator; i.e.,

$$\log \frac{M}{N} = \log M - \log N.$$

Proof. Let $x = \log_b M$ and $y = \log_b N$.

Express in exponential form: $M = b^x$ and $N = b^y$.

Divide the equations: $\dfrac{M}{N} = \dfrac{b^x}{b^y} = b^{x-y}$.

Change to logarithmic form: $\log_b \dfrac{M}{N} = x - y$.

$$\log_b \frac{M}{N} = \log_b M - \log_b N.$$

* The base is the same for all the logarithms.

Illustrations.

$$\log \frac{3}{7} = \log 3 - \log 7.$$

$$\log \frac{14}{15} = \log \frac{2 \cdot 7}{3 \cdot 5} = \log 2 + \log 7 - (\log 3 + \log 5).$$

Property 3. The logarithm of the **nth** *power of a number is equal to n times the logarithm of the number; i.e.,*

$$\mathbf{\log M^n = n \log M.}$$

Proof. Let	$x = \log_b M.$
Express in exponential form:	$M = b^x.$
Raise to the nth power:	$M^n = (b^x)^n = b^{nx}.$
Change to logarithmic form:	$\log_b M^n = nx.$
	$\log_b M^n = n \log_b M.$

Illustrations.

$$\log 16 = \log 2^4 = 4 \log 2.$$

$$\log \sqrt{7} = \log 7^{\frac{1}{2}} = \tfrac{1}{2} \log 7.$$

$$\log \frac{125}{49} = \log \frac{5^3}{7^2} = \log 5^3 - \log 7^2 = 3 \log 5 - 2 \log 7.$$

Note. Since $\sqrt[r]{M} = M^{\frac{1}{r}}$, $\log \sqrt[r]{M} = \frac{1}{r} \log M.$

Exercise 75

Given $\log_{10} 2 = 0.30$, $\log_{10} 3 = 0.48$, $\log_{10} 7 = 0.85$, *find the following logarithms.* (Remember $\log_{10} 10 = 1$.)

1. $\log_{10} 6.$ **2.** $\log_{10} 21.$

3. $\log_{10} \frac{3}{7}.$ **4.** $\log_{10} \frac{7}{2}.$

5. $\log_{10} 64.$ **6.** $\log_{10} 243.$

7. $\log_{10} \sqrt[5]{2}.$ **8.** $\log_{10} \sqrt[4]{3}.$

9. $\log_{10} \frac{14}{9}.$ **10.** $\log_{10} \frac{21}{32}.$

11. $\log_{10} 7000.$ **12.** $\log_{10} 300.$

13. $\log_{10} \sqrt{42}.$ **14.** $\log_{10} .02.$

15. $\log_{10} 5.$ **16.** $\log_{10} \sqrt[3]{\frac{7}{5}}.$

Express as a single logarithm. (Assume all logarithms have the same base.)

17. $\log(x-1) + \log(x+1)$.

18. $\log(3x-2) + \log 2$.

19. $\log(a+b) - \log b$.

20. $\log r - \log s - \log t$.

21. $7\log x + 5\log y - 3\log z$.

22. $2\log a + \log b - \log c - 5\log d$.

23. $\log 2 + \log \pi + \frac{1}{2}(\log l - \log g)$.

24. $\frac{1}{3}[\log x + \log y^2 - \log z]$.

Identify as true or false and give reasons. (In each equation the base is assumed to be the same for all logarithms.)

25. $\log\frac{1}{a} = -\log a$.

26. $\log\sqrt{a} = 2\log a$.

27. $\sqrt{a} = \frac{1}{2}\log a$.

28. $\frac{r}{s} = \log r - \log s$.

29. $(\log a)^3 = 3\log a$.

30. $\frac{1}{3}\log x + \frac{1}{2}\log y = \frac{1}{6}\log x^2y^3$.

31. $10^{6\log_{10} a} = a^6$.

32. $\log a = \frac{1}{n}\log a^n$.

33. $\log\frac{ab}{c} = \frac{\log a + \log b}{\log c}$.

34. $\log a^4 = 4\log\log a$.

142. Systems of logarithms. There are only two important systems of logarithms. The *natural*, or *Naperian*, system, uses the base e, where e is approximately 2.71828. This system is very useful in calculus and higher mathematics. The *common*, or *Briggs*, system employs the base 10. This system is most convenient for computation because our number system uses the base 10. Henceforth, in this book, when the base of a logarithm is not specified, we are to understand that the base is 10. Thus $\log N$ means $\log_{10} N$. Unless stated to the contrary, the word "logarithm" shall mean common logarithm.

143. Characteristic and mantissa. We know that

$$\begin{array}{rcl}
\log 1000 = 3 & \text{because} & 10^3 = 1000 \\
\log 100 = 2 & \text{because} & 10^2 = 100 \\
\log 10 = 1 & \text{because} & 10^1 = 10 \\
\log 1 = 0 & \text{because} & 10^0 = 1 \\
\log .1 = -1 & \text{because} & 10^{-1} = .1 \\
\log .01 = -2 & \text{because} & 10^{-2} = .01 \\
\log .001 = -3 & \text{because} & 10^{-3} = .001.
\end{array}$$

It seems reasonable to assume that if a number increases, its logarithm increases.* Consequently, any number between 10 and 100 must have a logarithm between 1 and 2. This logarithm can be written in the form 1 plus a positive decimal. For example, $\log 52.4 = 1 + .7193 = 1.7193$. Likewise, any number between .001 and .01 must have a logarithm between -3 and -2. This logarithm can be written in the form -3 plus a positive decimal. For example, $\log .004 = -3 + .6021$, which can be written $\log .004 = 7.6021 - 10$, because $-3 = 7 - 10$.

Every logarithm can be written as the sum of an integer (positive, negative, or zero) *plus a positive decimal.*† The integer is called the **characteristic** of the logarithm; the positive decimal is called the **mantissa** of the logarithm.

Illustrations.

Logarithm	Characteristic	Mantissa
3.4567	3	.4567
0.1234	0	.1234
2.0000	2	.0000
$7.8899 - 10$ or $-3 + .8899$	$7 - 10$ or -3	.8899
$8.4444 - 10$ or -1.5556	$8 - 10$ or -2	.4444

144. Method of determining characteristics. *If a number has a decimal point immediately to the right of its first nonzero digit, then the decimal point is said to be in* **standard position.** For example, the decimal point is in standard position in each of the following numbers: 2.003, 4.56, 3.2, and 7. Consequently, *if a number N has its decimal point in standard position*, then N is between 1 and 10, and $\log N$ is between 0 and 1; therefore *the characteristic of $\log N$ is* 0.

Theorem 1. Whenever a number is multiplied by 10, its logarithm is increased by 1.

Proof. Let $\log N$ be the logarithm of any number N. Then

$$\log 10N = \log 10 + \log N, \qquad \text{(Property 1)}$$
$$\log 10N = 1 + \log N.$$

* This is proved in more advanced books.

† Positive decimal here means a number n such that $0 \leq n < 1$.

Hence we see that when a number is multiplied by 10 (i.e., if the decimal point is moved one place to the right), the characteristic of its logarithm is increased by 1, but the mantissa is unaltered.

Illustration. If $\log 6.789 = 0.8318$, then $\log 67.89 = 1.8318$.

By repeating this process, we see that if a number is *multiplied* by 10^k (i.e., if the decimal point is moved k places to the right), the characteristic of its logarithm is *increased* by k. It also follows that if a number is *divided* by 10^k (i.e., if the decimal point is moved k places to the left), the characteristic of its logarithm is *decreased* by k.

Illustration. If $\log 6.789 = 0.8318$, then $\log 678.9 = 2.8318$, and $\log 67890 = 4.8318$, and $\log 0.6789 = 9.8318 - 10$, and $\log 0.006789 = 7.8318 - 10$.

We may sum up our discussion in the following

Theorem 2. If the decimal point in a number is k places to the $\begin{Bmatrix}\text{right}\\ \text{left}\end{Bmatrix}$ of standard position, then the characteristic of the logarithm of the number is $\begin{Bmatrix}k\\ -k\end{Bmatrix}$.

Illustration. The characteristic of $\log 7654$ is 3 because the decimal point is understood to be after the 4, which is 3 places to the right of standard position (i.e., after the 7).

The characteristic of $\log 0.07654$ is -2 or $8 - 10$ because the decimal point is 2 places to the left of standard position.

An alternate method used in finding characteristics is:

1. For a number N that is greater than 1, the characteristic is one less than the number of digits to the left of the decimal point in N.

2. For a number N that is less than 1, the characteristic is negative and is numerically equal to one more than the number of zeros between the decimal point and the first nonzero digit in N.

Illustrations.

Number	Characteristic of Logarithm
678900	5
678.9	2
6.789	0
0.6789	$9 - 10$ or -1
0.06789	$8 - 10$ or -2
0.0006789	$6 - 10$ or -4

Theorem 3. The mantissa of the logarithm of a number N is independent of the position of the decimal point in N. This means that two numbers differing only in the position of the decimal point have logarithms with the same mantissa. The proof of this theorem is embodied in that of theorem 1 and the discussion that follows it.

Illustration. The following numbers have logarithms with the same mantissa:

$$.00246, \quad 2.46, \quad 24.6, \quad 2460.$$

Theorem 2 serves two purposes:

(1) If we look at the position of the decimal point in a *number*, we can determine the characteristic of its *logarithm.*

(2) If we look at the characteristic of the *logarithm* of a number, we can determine the position of the decimal point in the *number.*

Example 1. Given $\log 7.05 = 0.8482$.

(*a*) Find $\log 0.00705$.

(*b*) Find N if $\log N = 2.8482$.

(*c*) Find N if $\log N = 6.8482 - 10$.

Solution.

(*a*) $\log 0.00705 = 7.8482 - 10$. Since the decimal point in 0.00705 is 3 places to the *left* of standard position, the characteristic is $-3 = 7 - 10$. The mantissa is the same as that for $\log 7.05$.

(*b*) If
$$\log N = 2.8482,$$
$$N = 705.$$
Since $\log N$ and $\log 7.05$ have the same mantissa, N is obtainable from 7.05 by moving the decimal point 2 places to the right (from standard position).

(*c*) If
$$\log N = 6.8482 - 10,$$
$$N = 0.000705.$$
Since the characteristic of $\log N$ is -4, we obtain N from 7.05 by moving the decimal point 4 places to the left.

Note. Theorems 1, 2, 3 of this article are valid only if the base of the logarithms is 10. For any other base, the process of finding the characteristic would not be so simple.

Exercise 76

Given $\log 6.65 = 0.8228$ *and* $\log 8.71 = 0.9400$, *find the value of the following.*

1. $\log 665$.
2. $\log 0.871$.
3. $\log 0.0871$.
4. $\log 0.00665$.
5. $\log 0.000871$.
6. $\log 66.5$.
7. $\log 6650$.
8. $\log 87100$.
9. $\log 0.665$.
10. $\log 871000$.
11. $\log 8710000$.
12. $\log 0.0000665$.

Given $\log 2.63 = 0.4200$ *and* $\log 83.0 = 1.9191$, *find* N *for each of the following.*

13. $\log N = 1.4200$.
14. $\log N = 3.9191$.
15. $\log N = 8.9191 - 10$.
16. $\log N = 9.4200 - 10$.
17. $\log N = 7.4200 - 10$.
18. $\log N = 6.4200 - 10$.
19. $\log N = 4.4200$.
20. $\log N = 2.9191$.
21. $\log N = 9.9191 - 10$.
22. $\log N = 0.9191$.
23. $\log N = 6.9191$.
24. $\log N = 9.4200 - 20$.
25. $\log N = -0.5800$.
26. $\log N = -0.0809$.
27. $\log N = -2.0809$.
28. $\log N = -1.5800$.

Hint. For Probs. 25 to 28, first write $\log N$ in the form where the decimal part is positive. For example, $-0.5800 = 9.4200 - 10$.

145. A four-place table of mantissas. In Table III (page 310) there are listed, to four decimal places, the mantissas (with the decimal points omitted) of the logarithms of all positive integers from 1 to 999. Since the mantissa is independent of the decimal point in the number, Table III can be used to find the mantissa of the logarithm of any three-figure number. The problems we shall need to consider are:

1. Given a number N, to find $\log N$.
2. Given $\log N$, to find N.

146. Given N, to find $\log N$. The procedure of finding the logarithm of a given number is illustrated by the following example.

Example 1. Find $\log 0.0526$.

Solution. The characteristic of $\log 0.0526$ is -2 or $8 - 10$. To find the mantissa, look for 52 in the left-hand column headed N on page 310. In the line beginning with 52, move over to the column headed by 6 and find 7210. Hence

$$\log 0.0526 = 8.7210 - 10.$$

147. Given log N, to find N. The procedure of finding the number whose logarithm is given is illustrated by the following examples.

Example 1. Given $\log N = 1.3345$, find N.

Solution. Look for the mantissa .3345 in the body part of Table III. It appears in the 21 line and the 6 column. Hence N is 216 with the decimal point placed in accordance with a characteristic of 1. Therefore,

if
$$\log N = 1.3345,$$
$$N = 21.6.$$

Example 2. Given $\log N = 7.0969 - 10$, find N.

Solution. The mantissa .0969 appears in the 12 row and the 5 column. Hence N is 125 with the decimal point moved 3 places to the left of standard position (because the characteristic is $7 - 10 = -3$). Therefore,

if
$$\log N = 7.0969 - 10,$$
$$N = 0.00125.$$

Exercise 77

Use a four-place table to find the value of each of the following.

1. log 87.6.

2. log 3450.

3. log 0.712.

4. log 0.0946.

5. log 0.00209.

6. log 0.000831.

7. log 447000.

8. log 56200.

9. log 1.04.

10. log 0.620.

Use a four-place table to determine the value of N in each of the following.

11. $\log N = 2.5211$.

12. $\log N = 1.8837$.

13. $\log N = 8.7348 - 10$.

14. $\log N = 9.3909 - 10$.

15. $\log N = 6.9509 - 10$.

16. $\log N = 7.1492 - 10$.

17. $\log N = 0.0453$.

18. $\log N = 3.7959$.

19. $\log N = 9.4440 - 10$.

20. $\log N = 0.6693$.

21. $\log N = 9.8035 - 10$.

22. $\log N = 8.4609 - 10$.

148. Interpolation. When a sports announcer says, "The ball is on the 27-yard line," most football fans realize that the announcer *estimates* that the ball is $\frac{2}{5}$ of the way from the 25-yard line to the 30-yard line. This process of literally "reading between the lines" is called **interpolation.** Another example is, "Interpolate to approximate the value of $\sqrt{8}$." Knowing $\sqrt{4} = 2$ and $\sqrt{9} = 3$, we conclude that $\sqrt{8}$ is a number between 2 and 3. Moreover, 8 is $\frac{4}{5}$ of the way from 4 to 9. Assume that for a small increase in a number N, the change in $\sqrt{N}$ is proportional to the change in N.* Then $\sqrt{8}$ would be $\frac{4}{5}$ of the way from 2 to 3. Since $\frac{4}{5}$ of 1 is .8, we conclude that $\sqrt{8}$ is approximately 2.8. This result is correct to only one decimal place. The process of interpolation is important in all work involving the use of tables.

We have seen that the logarithm of a three-figure number can be found by use of Table III. If a number is increased by a small amount, the change in its logarithm is very nearly proportional to the change in the number itself. Hence the logarithm of a four-figure number can be found from Table III by interpolation. It is to be remembered that, since the mantissas of the logarithms of most integers are unending decimals, a four-place table merely rounds off these mantissas to four-figure accuracy. Hence the results obtained in using a four-place table will usually be only approximations with the last figure subject to error.

Example 1. Find log 182.7.

Solution. The characteristic is 2. Since the mantissa of the logarithm is independent of the position of the decimal point in the number, we see that the required mantissa lies between the mantissas for 1820 and 1830.

$$\left.\begin{array}{l}\left.\begin{array}{l}\text{mantissa of log}\dagger\ 1820\\ \text{mantissa of log}\ 1827\end{array}\right\}7\\ \text{mantissa of log}\ 1830\end{array}\right\}10 \quad \left.\begin{array}{l}=.2601\\ =\\ =.2625\end{array}\right\}24$$

As the number increases by 10 (from 1820 to 1830), the mantissa increases 24 ten-thousandths. Our number is $\frac{7}{10}$ of the way from

* Not strictly true, but a good approximation if the change is small.

† A convenient abbreviation for *mantissa of log* is *m-log*.

1820 to 1830. Hence the required mantissa is $\frac{7}{10}$ of the way from .2601 to .2625. But $\frac{7}{10}(24) = 16.8 \rightarrow 17$. (Round off to 17 because 16.8 * is closer to 17 than it is to 16.) Add the 17 ten-thousandths to .2601 to get .2618, the required mantissa. Hence

$$\log 182.7 = 2.2618.$$

Example 2. Given $\log N = 9.2950 - 10$, find N.

Solution. We search for the mantissa .2950 in the body part of Table III. It lies between the mantissas .2945 and .2967.

$$\left.\begin{array}{l} m\text{-}\log 1970 \\ m\text{-}\log N \\ m\text{-}\log 1980 \end{array}\right\} 10 \quad \begin{array}{l} = .2945 \\ = .2950 \\ = .2967 \end{array} \quad \begin{array}{l} \}5 \\ \\ \end{array} \quad 22$$

The given mantissa is $\frac{5}{22}$ of the way from .2945 to .2967. Hence the required number (aside from decimal point) should be $\frac{5}{22}$ of the way from 1970 to 1980. But $\frac{5}{22}(10) = 2\frac{3}{11} \rightarrow 2$. Adding 2 to 1970, we find N is 1972 with the decimal point placed in accordance with a characteristic of $9 - 10$. Hence

if
$$\log N = 9.2950 - 10,$$
$$N = 0.1972.$$

Exercise 78

Use a four-place table to find the value of the following.

1. log 5.437.
2. log 32.18.
3. log 0.04153.
4. log 0.7281.
5. log 0.0008905.
6. log 0.008607.
7. log 271.6.
8. log 55620.
9. log 0.1472.
10. log 0.02244.
11. log 123100.
12. log 6429.
13. log 700.4.
14. log 4.956.
15. log 0.009469.
16. log 0.0001733.

* In "rounding off" a number that is *exactly halfway*, it is conventional to choose the number that makes the *final result* of the interpolation *even* rather than odd. This procedure will be followed throughout this book.

Use a four-place table to determine the value of N in each of the following.

17. $\log N = 1.4810.$

18. $\log N = 0.9743.$

19. $\log N = 9.3826 - 10.$

20. $\log N = 8.2921 - 10.$

21. $\log N = 7.6543 - 10.$

22. $\log N = 6.1771 - 10.$

23. $\log N = 2.7441.$

24. $\log N = 3.5493.$

25. $\log N = 4.0913.$

26. $\log N = 5.7806.$

27. $\log N = 8.9343 - 10.$

28. $\log N = 9.8766 - 10.$

29. $\log N = 5.2607 - 10.$

30. $\log N = 7.0328 - 10.$

31. $\log N = 0.1874.$

32. $\log N = 1.2252.$

149. Approximations and significant figures. If a given distance is measured and if its length is expressed in decimal form, it is conventional to write no more digits than are correct (or probably correct). Thus if we say that the measured distance between points A and B is 17 feet, we mean that the result is given to the nearest foot, i.e., the true distance is closer to 17 feet than it is to 16 feet or 18 feet. This is an example of two-figure accuracy. If we say that the measured distance AB is 17.0 feet, we mean that the true distance is given to three significant figures, i.e., it is closer to 17.0 than it is to 16.9 or 17.1. This implies that the true distance is somewhere between 16.95 and 17.05. Notice that 17 and 17.0 do not mean the same thing when they represent approximate values.

The number of significant digits in a number is obtained by counting the digits from left to right, beginning with the first nonzero digit and ending with the rightmost digit.* Thus, 0.078060 has five significant digits, 70.00 has four, and 0.790 has only three. Notice that the number of significant digits does not depend on the position of the decimal point.

Results computed by multiplication or division from approximate data will usually have no higher degree of accuracy than that of the

* Ambiguity may result if the number in question is an *integer* ending in one or more 0's. For example, if the radius of the earth is given as 4000 miles, we may not know how many 0's are significant. If, however, the number 4000 was obtained from 3960 by rounding it off to the nearest multiple of 100 miles, then the first 0 is significant; the other two are not. In a case of this kind, computers sometimes underscore the digits that are significant: $\underline{40}00$. Another way of indicating that 4000 represents two-figure accuracy is to express it as $4.0(10^3)$.

data used. *We agree to round off the result so that it will have as many significant figures as there are in the least accurate number in the data.* If a field is measured and found to be 11.3 rods long and 10.7 rods wide, we would be tempted to say that its area is $(11.3)(10.7) = 120.91$ square rods. To do so would be to claim false accuracy. The result should be rounded off to three significant figures (the same as in the given data) to obtain 121 square rods. The first two figures in this result are correct but the third is only a good approximation because the true area is somewhere between $(11.25)(10.65) = 119.8125$ square rods and $(11.35)(10.75) = 122.0125$ square rods.

In using addition and subtraction to compute results from approximate data, do not retain a given place in the result unless this place is significant in each of the given data. For example, if we add the three approximate values $a = 5.12$, $b = 13.0078$, $c = 1021.009$, we obtain $a + b + c = 1039.1368 \rightarrow 1039.14$. The result was rounded off to two-decimal place accuracy because the value of a is written to only two-decimal accuracy. Hence the result is not reliable beyond the second decimal place.

Since nearly all logarithms of rational numbers are irrational, the mantissas given in Table III are merely four-figure approximations. Hence most of the results obtained by use of logarithms will be approximations and should be considered as such. In no case can an accuracy of more than 4 significant figures be obtained in using a 4-place table. If 5 significant figures are required, one should use a 5-place table, or preferably a 6-place table.

150. Logarithmic computation. When logarithms are used to compute products, quotients, and powers of numbers, it is advisable to:

1. Make a complete outline indicating the operations to be performed.
2. Fill in all characteristics.
3. Fill in all mantissas.
4. Perform the operations outlined in Step 1.

These suggestions are offered in the hope that accuracy, speed, and neatness will be achieved. Every logarithm appearing in the solution should be labeled.

Example 1. Use logarithms to compute $\dfrac{(2460)(0.357)}{8.18}$.

Solution. Let
$$N = \frac{(2460)(0.357)}{8.18}.$$

Then $\log N = \log 2460 + \log 0.357 - \log 8.18$. After preparing the outline and filling in the characteristics, we have

$$\begin{aligned}
\log 2460 &= 3. \\
\log 0.357 &= \underline{9. \qquad - 10} \text{ Add} \\
\log \text{numerator} &= \\
\log 8.18 &= \underline{0. \qquad\qquad} \text{ Subtract} \\
\log N &= \\
N &=
\end{aligned}$$

After supplying the mantissas and performing the indicated operations, we have

$$\begin{aligned}
\log 2460 &= 3.3909 \\
\log 0.357 &= \underline{9.5527 - 10} \text{ A} \\
\log \text{numerator} &= 12.9436 - 10 \\
\log 8.18 &= \underline{0.9128 \qquad\quad} \text{ S} \\
\log N &= 2.0308 \\
N &= 107.
\end{aligned}$$

Notice that no interpolation was performed in finding N from $\log N$. The original numbers are all three-figure numbers; hence the computed result should have no more than three-figure accuracy. Since the mantissa .0308 is best approximated by the tabular mantissa .0294, the best three-figure approximation of N is 107.

In all logarithmic computations we are really expressing the original numbers as powers of 10. For example, since $\log 2460 = 3.3909$, it follows that $2460 = 10^{3.3909}$. Consequently

$$N = \frac{(10^{3.3909})(10^{9.5527-10})}{10^{0.9128}}.$$

Apply the laws of exponents:

$$\begin{aligned}
N &= 10^{3.3909+(9.5527-10)-0.9128} \\
&= 10^{2.0308} = 107.
\end{aligned}$$

Example 2. Use logarithms to compute

$$N = \frac{(1.789)^3}{(87650)(0.04466)}.$$

Solution. Take the logarithm of each side:

$$\log N = 3 \log 1.789 - (\log 87650 + \log 0.04466).$$

After taking the four suggested steps, we get

$$\begin{array}{rll@{\qquad}rll}
\log 1.789 = & \underline{0.2526} & 3 & \log 87650 = & 4.9428 & \\
3 \log 1.789 = & 0.7578 & & \log 0.04466 = & \underline{8.6499 - 10} & \text{A} \\
\log \text{num.} = & 10.7578 - 10 & & \log \text{den.} = & 13.5927 - 10 & \\
\log \text{den.} = & \underline{3.5927} & \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- & = & 3.5927 & \\
\log N = & 7.1651 - 10 & \text{S} & & & \\
N = & 0.001462 & & & &
\end{array}$$

Notice that log num. was changed from 0.7578 to 10.7578 − 10 to avoid subtracting 3.5927 from a smaller number.

This example illustrates four-figure accuracy in the original data and in the computed result.

Example 3. Compute $\sqrt[3]{\dfrac{0.00559}{90.16}}$.

Solution. Let $\qquad N = \sqrt[3]{\dfrac{0.00559}{90.16}}.$

Then $\qquad \log N = \frac{1}{3}[\log 0.00559 - \log 90.16].$

After taking the four suggested steps, we have

$$\begin{array}{rll}
\log 0.00559 = & 7.7474 - 10 & \\
\log 90.16 = & \underline{1.9550} & \text{S} \\
\log \text{radicand} = & 5.7924 - 10 & \\
\log \text{radicand} = & 25.7924 - 30 & \\
3 & \overline{} & \\
\log N = & 8.5975 - 10 & \\
N = & 0.0396. &
\end{array}$$

Notice that log radicand was changed from 5.7924 − 10 to 25.7924 − 30 to facilitate the division by 3. Had we divided 5.7924 − 10 by 3, the result, 1.9308 − 3.3333 = −1.4025, would

involve a *negative* decimal that does not occur in our table of *positive* mantissas.

The final result is written with only three-figure accuracy because the least accurate number, 0.00559, in the original data has only three significant figures.

Example 4. Use logarithms to compute

$$x = \frac{(-1.789)^3}{(-87650)(-0.04466)}.$$

Solution. The value of x is negative since $\dfrac{(-)^3}{(-)(-)} = \dfrac{-}{+} = -.$

Discard all minus signs in x and then use logarithms to compute the value of the corresponding expression in which all numbers are positive. This was done in Ex. 2 with the result 0.001462. Hence

$$x = -0.001462.$$

Exercise 79

Use logarithms to compute the following correct to three-figure accuracy. (In finding N from $\log N$, do not interpolate.)

1. $(35.7)(4.68)$.

2. $(872)(2.09)(0.00685)$.

3. $(0.493)^2(1950)$.

4. $(0.0841)(6.18)$.

5. $\dfrac{612}{0.457}$.

6. $\dfrac{478}{2.69}$.

7. $\dfrac{\sqrt{7.08}}{29.3}$.

8. $\dfrac{0.00582}{(7.34)^2}$.

9. $(0.975)^{100}$.

10. $(3.00)^{20}$.

11. $[0.234)(5.19)]^6$.

12. $\left(\dfrac{7.84}{12.9}\right)^3$.

13. $\dfrac{(1620)(342)}{0.0725}$.

14. $\dfrac{(0.0879)(62.4)}{(3.14)(9060)}$.

15. $\dfrac{(5.03)^7}{8270}$.

16. $(2.86)\sqrt{0.948}$.

17. $\sqrt[3]{0.0000627}$.

18. $\sqrt[3]{\dfrac{0.00158}{926}}$.

19. $\sqrt[7]{(0.0391)(1.68)}$.

20. $\sqrt[6]{0.0432}$.

21. $\left[\dfrac{-3.07}{(-0.894)(-623)}\right]^5$.

22. $\sqrt[9]{\dfrac{(-27.1)(-9460)}{-548}}$.

23. $\sqrt[3]{\dfrac{(-473)^2}{(-7.58)(83.6)}}$.

24. $\left[\dfrac{(-146)\sqrt[3]{-92.5}}{-4350}\right]^7$.

25. $(70.4)^{\frac{3}{7}}$.

26. $(0.238)^{-9}$.

27. $\dfrac{(2.13)^8 + 389}{74.1}$.

28. $\dfrac{\sqrt[6]{209} - 3.79}{0.451}$.

29. $\dfrac{\log 80.2}{\log 3.02}$.

30. $(\log 796)(\log 50.2)$.

31. $(\log 912)^3(6.23)$.

32. $(\log 5.28)\sqrt{69.3}$.

Use logarithms to compute the following correct to four-figure accuracy.

33. $(1947)(0.8263)$.

34. $(520300)(0.007742)$.

35. $\dfrac{99.44}{638.5}$.

36. $\dfrac{0.4038}{66.79}$.

37. $\dfrac{842.1}{\sqrt{3827}}$.

38. $\dfrac{(20.05)^3}{3202}$.

39. $\sqrt[3]{(0.4219)(0.07806)}$.

40. $\sqrt[7]{51640}$.

41. $(1.721)^4$.

42. $[(0.2438)(4.006)]^6$.

43. $\dfrac{92.37}{(5.260)^2}$.

44. $\dfrac{\sqrt{8765}}{9.264}$.

Use logarithms to compute the following to as much accuracy as is warranted by the numbers involved.

45. $(0.009263)(0.0473)$.

46. $\dfrac{7.16}{23.45}$.

47. $\dfrac{\sqrt{0.056}}{0.724}$.

48. $(2.3)^5(0.648)$.

49. $\dfrac{9.07}{(56200)(15.3)}$.

50. $581\sqrt{0.0764}$.

51. $(0.827)^5$.

52. $\sqrt[7]{0.000834}$.

53. $\sqrt[7]{\dfrac{0.00823}{4.261}}$.

54. $(89.2)^{\frac{4}{5}}$.

55. $\sqrt[3]{(0.07240)(32.10)}$.

56. $\sqrt[3]{\dfrac{1776}{(5.670)(89.26)}}$.

57. $[(0.4560)(8.234)]^6$.

58. $\left[\frac{0.9427}{1.030}\right]^7$.

59. The time t in seconds for one complete oscillation of a pendulum of length l feet is given by

$$t = 2\pi\sqrt{\frac{l}{g}},$$

where $g = 32.2$ feet per second per second. Find t for a pendulum 1.92 feet long. Use $\pi = 3.14$.

60. The radius r of the inscribed circle of a triangle whose sides are a, b, c is given by the formula

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$

where $s = \frac{1}{2}(a + b + c)$. Compute r when $a = 5.72$, $b = 6.39$, $c = 7.41$.

151. Exponential equations. An exponential equation is an equation in which the unknown appears in an exponent. Such an equation can usually be solved by equating the logarithms of the two sides and then finding the roots of the resulting algebraic equation.

Example 1. Solve for x: $(.195)^x = 26.8$.

Solution. Take the logarithm of each side:

$$\begin{aligned} \log (.195)^x &= \log 26.8 \\ x \log .195 &= \log 26.8 \\ x &= \frac{\log 26.8}{\log .195} \\ &= \frac{1.4281}{9.2900 - 10} \\ &= \frac{1.4281}{-.7100} = -2.01. \end{aligned}$$

In this case it is easier to perform the division without using logarithms. If logarithms are used, we should first compute the value of the fraction $\frac{1.4281}{.71}$ and then attach a minus sign to the result.

It should be noted that $\frac{\log 26.8}{\log .195}$ is the quotient of two logarithms and not the logarithm of a quotient.

Exercise 80

Solve for x.

1. $(63.1)^x = 789$.
2. $(3.09)^x = 2.28$.
3. $(.0912)^x = 23.1$.
4. $(.708)^x = 3630$.
5. $(8.11)^x = .567$.
6. $(26.3)^x = .0722$.
7. $(.302)^x = .00739$.
8. $(.00955)^x = .275$.
9. $(.00871)^{x-5} = 6.78$.
10. $(.0865)^{2x-3} = 10.6$.
11. $(5012)^{2x+7} = 7806$.
12. $(25120)^{x+1} = 456$.
13. $(7.92)^{x+1} = (1.58)^x(55.2)$.
14. $(1.91)^x(1.62)^{2x+1} = 348$.
15. $\dfrac{(1.04)^x - 1}{.04} = 3.25$.
16. $\dfrac{(1.03)^x - 1}{.03} = 2.29$.
17. $(4.13)^{x^2-6x} = 17.8$.
18. Solve for x and y:
$$\begin{cases} (2.27)^{x+y} = 81.6 \\ (49.2)^{2x-y} = .705. \end{cases}$$
19. If $S = \dfrac{a - ar^n}{1 - r}$, show that $n = \dfrac{\log (a - S + rS) - \log a}{\log r}$.
20. Solve for n: $A = P(1 + i)^n$.

Solve for y in terms of x.

21. $\log y - 3 \log x = 2 + \log 7$.
22. $\log y + \log 3 = x + 8$.
23. $\log_e y + 5x = \log_e 9$.
24. $\log y - \log x = 2$.

152. Change of base of logarithms. For the purpose of making numerical computation, the most convenient system of logarithms is the *common*, or *Briggs*, system, which employs the base 10. If we know the logarithm of a number to the base a, we can find the logarithm of that number to the base b by using

$$\log_b N = \frac{\log_a N}{\log_a b} = (\log_b a)(\log_a N). \tag{1}$$

To prove this, let $y = \log_b N$. Then

$$b^y = N.$$

Take the logarithm of each side to the base a:

$$\log_a b^y = \log_a N.$$
$$y \log_a b = \log_a N.$$
$$\log_b N \log_a b = \log_a N.$$

Hence $$\log_b N = \frac{\log_a N}{\log_a b}.$$

If $N = a$, $$\log_b a = \frac{1}{\log_a b}.$$

Therefore $$\log_b N = (\log_b a)(\log_a N).$$

In higher mathematics, the most suitable system of logarithms is the *natural*, or *Naperian*, system, which employs the base e, where e is an irrational number whose approximate value is 2.71828. If $a = 10$ and $b = e$, equation (1) becomes

$$\log_e N = \frac{\log_{10} N}{0.4343} = 2.3026 \log_{10} N.$$

Thus the natural logarithm of a number can be obtained by multiplying its common logarithm by 2.3026.

Exercise 81

Find the following logarithms.

1. $\log_e 63.1$.	**2.** $\log_e 631$.	**3.** $\log_e .0955$.
4. $\log_e .912$.	**5.** $\log_3 .101$.	**6.** $\log_5 .0398$.
7. $\log_2 64.9$.	**8.** $\log_7 350$.	**9.** $\log_{12} 2.63$.
10. $\log_3 \pi$.	**11.** $\log_{\frac{1}{9}} 730$.	**12.** $\log_{10} e$.

chapter 18 *Theory of investment*

153. Interest. Interest is money paid for the use of borrowed money called the **principal.** If the interest for a unit period of time (usually a year) is divided by the principal, the resulting fraction is called the **rate of interest** (usually expressed in per cent). The sum of the principal and the interest is called the **amount.**

154. Simple interest. Interest that is computed on only the original principal is called **simple interest.** Let P be the principal, i the rate of interest per year, n the number of years, I the interest, and A the amount. Then

$$I = Pni \tag{1}$$

and

$$A = P + I = P(1 + ni). \tag{2}$$

Example 1. Find the simple interest and the amount on \$700 for 8 months at 6%. (This conventionally means 6% per annum.)

Solution. We have $P = \$700, \quad n = \frac{2}{3}, \quad i = .06.$

Using (1), $\qquad I = \$700(\frac{2}{3})(.06) = \$28.$

Using (2), $\qquad A = \$700 + \$28 = \$728.$

155. Compound interest. If, at the end of a unit period of time, the interest is added to the principal to form a new principal for the next unit period of time, the interest is said to be **compounded.** The unit period of time is called the **conversion period.** It is usually a year, half year, or a quarter year. The sum of the principal and

interest at the end of a conversion period is called the **compound amount.**

Compound interest is usually quoted on a yearly rate. Thus, the rate 6% *compounded semiannually* means 3% for each half-year period. The *6%* figure is called the **nominal rate** of interest.

156. Compound interest formula. If a principal P is invested at an interest rate i per conversion period, then the compound amount A to which P accumulates at the end of n conversion periods is given by the formula

$$A = P(1 + i)^n.$$

Proof. At the end of the first conversion period the interest is Pi and the amount is $P + Pi$, or $P(1 + i)$. This is the new principal for the second period at the end of which the interest is $P(1 + i)$ times i, or $P(1 + i)i$. The amount at the end of the second period is $P(1 + i) + P(1 + i)i$, or $P(1 + i)^2$. We see that if the principal at the beginning of a period is multiplied by $(1 + i)$, we obtain the amount at the end of the period. At the end of n periods the original principal P will have been multiplied by $(1 + i)$ to n factors. Hence $A = P(1 + i)^n$.

Example 1. Find the compound amount on \$2500 at 6% compounded semiannually for $5\frac{1}{2}$ years.

Solution. Since the conversion period is $\frac{1}{2}$ year, we have $n = 11$ and $i = 3\%$. Hence

$$\begin{aligned} A = P(1 + i)^n &= \$2500(1.03)^{11} \\ &= \$2500(1.3842) \\ &= \$3460.50. \end{aligned}$$

The value of $(1.03)^{11}$ was found by use of Table IV. In the column headed 3%, in line with $n = 11$, we find 1.3842.

Comments. 1. If more accuracy is desired, we can compute $(1.03)^{11}$ by using the binomial theorem to expand $(1 + .03)^{11}$ to as many terms as are needed. In this case, six terms are necessary to obtain the result, \$3460.58, correct to the nearest cent.

2. Another method of computing $(1.03)^{11}$ is by logarithms. This is not advisable unless a six or seven-place table is used. A four-

place table gives a result that is less accurate than that obtained from Table IV.

157. Present value. Suppose that a sum of money A is due at some future date. The present value of A is the principal P which, if invested at interest, will accumulate to A at the end of the specified time. Let n represent the number of conversion periods and let i be the rate of interest per period. Using $A = P(1 + i)^n$, we find $P = \dfrac{A}{(1 + i)^n}$ or

$$P = A(1 + i)^{-n}.$$

Example 1. How much money must be placed in a savings account paying 2% compounded annually in order that this sum will accumulate to \$1000 in 18 years?

Solution.

$$\begin{aligned} P = A(1 + i)^{-n} &= \$1000(1.02)^{-18} \\ &= \$1000(.70016) \\ &= \$700.16. \end{aligned}$$

The value of $(1.02)^{-18}$ was found by use of Table V.

Exercise 82

Find the simple interest and the amount.

1. On \$1200 for 6 months at 4%.
2. On \$800 for 9 months at 6%.
3. On \$640 for 7 months at 3%.

Work the following simple interest problems.

4. What principal will amount to \$1000 at the end of 1 year at 5%?
5. How long will it take a principal of \$600 to amount to \$610 at 4%?
6. What rate of interest is needed to make a principal of \$1500 amount to \$1520 at the end of two months?

Find the compound amount and the compound interest.

7. On \$400 for 8 years at 5% compounded annually.
8. On \$700 for 10 years at 6% compounded semiannually.
9. On \$1000 for 7 years at 8% compounded quarterly.

10. On \$3000 for 12 years at $2\frac{1}{2}\%$ compounded annually.

11. Find the present value of \$500 due in 15 years if money is worth 4% compounded annually.

12. Find the present value of \$600 due in 5 years if the interest rate is 3% compounded semiannually.

13. A man wishes to leave his grandson the sum of \$10,000 on Jan. 10, 1970. How much should he deposit on Jan. 10, 1950 to the boy's account in a bank that pays 3% compounded semiannually?

14. What sum of money invested at 4% compounded annually will amount to \$4000 at the end of 18 years?

15. How long will it take a principal to double itself at 2% interest compounded annually? (Use Table IV and obtain result to the nearest year.)

16. How much compound interest will \$1000 earn in 80 years if invested at 5% compounded annually?
Hint. $(1.05)^{80} = [(1.05)^{40}]^2$.

17. Five years ago a banker lent \$6000 to a businessman. Three years ago the banker lent another \$4000 to the same man. Two years ago the man made a payment of \$5000. How much should he pay the banker now to discharge his obligation if the interest rate is 5% compounded annually?

18. The interest rate "8% compounded quarterly" is equivalent to what interest rate * compounded annually?

Solve by use of logarithms.

19. At what rate of interest compounded annually will \$100 amount to \$131.70 at the end of 8 years?

20. How long will it take a principal to double itself at 8% interest compounded annually?

158. Annuities. An **annuity** is a sequence of equal payments of money made at equal intervals of time. For example, the annual payments on a life insurance policy represent an annuity. Likewise, the monthly payments on a piano or refrigerator constitute an annuity. The length of time between successive payments is called the **payment period.** We shall consider only the ordinary annuity in which a payment is made at the *end* of each period. The **term** of an annuity is the length of time between the beginning of the first payment period and the end of the last payment period.

* This rate is called the **effective rate,** whereas 8% is called the **nominal rate.**

159. Amount of an annuity. The **amount of an annuity** is the sum of the *compound amounts* of the various payments accumulated at the end of the term. It is understood that the interest is compounded whenever a payment is made, i.e., at the *end* of each payment period.

Illustration. An annuity of \$100 a year for 3 years consists of a payment of \$100 at the end of a year, another payment of \$100 at the end of two years, and a final payment of \$100 at the end of three years. If the interest rate is 6%, the amount of this annuity (i.e., its accumulated value at the end of 3 years) is

$$\begin{aligned} & 100(1.06)^2 + 100(1.06) + 100 \\ = {} & 112.36 + 106 + 100 \\ = {} & 318.36. \end{aligned}$$

We shall now develop a formula for the amount of an annuity of \$1 per period for n periods, with interest rate i per period. The first payment which is made at the end of the first period, will draw interest for $(n - 1)$ periods and consequently will amount to $(1 + i)^{n-1}$. Similarly, the second payment will accumulate to $(1 + i)^{n-2}$. The next to the last payment, which draws interest for only one year, will amount to $(1 + i)$. The last payment, which is made at the end of the last period, will amount to \$1. Let $s_{\overline{n}|}$ (read "s angle n") represent the sum of these amounts (which we write in reverse order):

$$s_{\overline{n}|} = 1 + (1 + i) + \cdots + (1 + i)^{n-2} + (1 + i)^{n-1}.$$

The right side of this equation is a geometric progression of n terms with the common ratio $(1 + i)$. Using the formula $S = \dfrac{a - rl}{1 - r}$ and then simplifying, we get

$$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}.$$

If each payment is \$$R$ instead of \$1, the amount of the annuity is $Rs_{\overline{n}|}$:

$$\textbf{Amount of annuity} = S = Rs_{\overline{n}|} = R\,\frac{(1 + i)^n - 1}{i}.$$

The value of $s_{\overline{n}|}$ for various rates of interest may be found in Table VI.

Example 1. A man deposits \$200 in a savings bank at the end of each half year. Find the amount to his account at the end of 12 years if the interest rate is 5% compounded semiannually.

Solution. The payment period is $\frac{1}{2}$ year. Hence $n = 24$ and $i = 2\frac{1}{2}\%$. The amount of the annuity is

$$\begin{aligned} S &= \$200(s_{\overline{24}|} \text{ at } 2\tfrac{1}{2}\%) \\ &= \$200(32.3490) \\ &= \$6469.80. \end{aligned}$$

160. Present value of an annuity. The **present value of an annuity** is the sum of the present values of the various payments.

Illustration. The present value of an annuity of \$100 per year for 3 years, money being worth 6% per annum, is

$$\begin{aligned} &100(1.06)^{-1} + 100(1.06)^{-2} + 100(1.06)^{-3} \\ &= 94.340 + 89.000 + 83.962 \\ &= \$267.30. \end{aligned}$$

We shall now derive a formula for the present value of an annuity of \$1 per period for n periods, with interest rate i per period. The first payment is made at the end of the first period. Its present value is $(1 + i)^{-1}$. Similarly, the present value of the second payment is $(1 + i)^{-2}$. The final payment is made at the end of the nth period. Its present value is $(1 + i)^{-n}$. Let $a_{\overline{n}|}$ represent the present value of the annuity. Then

$$a_{\overline{n}|} = (1 + i)^{-1} + (1 + i)^{-2} + \cdots + (1 + i)^{-n}.$$

The right side of this equation is a geometric progression of n terms with $(1 + i)^{-1}$ as the first term and the common ratio. Using the formula $S = \dfrac{a - rl}{1 - r}$, we get

$$a_{\overline{n}|} = \frac{(1 + i)^{-1} - (1 + i)^{-n-1}}{1 - (1 + i)^{-1}}.$$

If top and bottom of the right side are multiplied by $(1 + i)$, we get

$$a_{\overline{n}|} = \frac{1 - (1 + i)^{-n}}{i}.$$

If each payment is $R instead of $1, the present value of the annuity is $Ra_{\overline{n}|}$:

$$\textbf{Present value of annuity} = A = Ra_{\overline{n}|} = R\,\frac{1-(1+i)^{-n}}{i}.$$

The value of $a_{\overline{n}|}$ for various rates of interest may be found in Table VII.

As a check, the present value of the $1 annuity is the present value of the amount of the annuity. Since the amount $s_{\overline{n}|}$ is due at the end of n periods, we multiply $s_{\overline{n}|}$ by $(1+i)^{-n}$ to get $a_{\overline{n}|}$. Hence

$$\begin{aligned} a_{\overline{n}|} &= s_{\overline{n}|}(1+i)^{-n} = \frac{(1+i)^n - 1}{i}(1+i)^{-n} \\ &= \frac{1-(1+i)^{-n}}{i}. \end{aligned}$$

Example 1. What sum of money should be invested at $2\frac{1}{2}\%$ compounded annually to yield $500 at the end of each year for 10 years?

Solution. We are to find the present value of the annuity.

$$\begin{aligned} A = Ra_{\overline{n}|} &= \$500(a_{\overline{10}|} \text{ at } 2\tfrac{1}{2}\%) \\ &= \$500(8.7521) \\ &= \$4376.05. \end{aligned}$$

Example 2. A man buys a house for $8000. He pays $3000 cash and agrees to pay the remainder, including interest at 6% compounded annually, in 15 equal annual installments. How large is each annual payment?

Solution. Let $R be the annual payment. Then $5000 represents the present value of an annuity of $R per year for 15 years. Hence

$$5000 = R(a_{\overline{15}|} \text{ at } 6\%).$$

$$R = \frac{5000}{a_{\overline{15}|} \text{ at } 6\%} = \frac{5000}{9.7122} = 514.82.$$

Exercise 83

Find the amount and the present value of the following annuities.

1. $120 annually for 40 years at 3% compounded annually.
2. $300 semiannually for 10 years at 4% compounded semiannually.

3. \$50 quarterly for 12 years at 8% compounded quarterly.

4. \$600 annually for 30 years at 4% compounded annually.

5. A boy deposits \$100 in a savings bank at the end of each year. What will his savings amount to at the end of 5 years if the interest rate is $2\frac{1}{2}\%$ compounded annually?

6. In order to finance a college education for his son, a man deposits \$200 each year for 16 years in a bank which pays 3% compounded annually. If the first payment is made on the boy's 3rd birthday, how much money will he have on his 18th birthday?

7. What sum of money should be invested at 3% compounded semiannually to yield \$100 at the end of each half year for $8\frac{1}{2}$ years?

8. A man buys a piece of property and makes a down payment of \$1000. He agrees to pay \$500 annually at the end of each of the next four years. If money is worth 5% compounded annually, find the equivalent cash price.

9. What sum of money should a philanthropist deposit in a bank paying 3% compounded annually if five annual scholarships of \$200 each are to be made available for a period of 30 years? The scholarships are to begin 1 year from now.

10. If money is worth 4% compounded annually, which of the following annuities has the larger cash value? First, ten annual payments of \$100 each, the first to be made at the end of 1 year. Second, eight annual payments of \$120 each, the first to be made now.

11. A man wants to give his son \$5000 on his 18th birthday. Each year he will deposit a given sum of money to the boy's account with a trust company that pays $2\frac{1}{2}\%$ compounded annually. What sum must be deposited each year if the first of 18 annual payments is made on the boy's first birthday?

12. Every 5 years a utility company must buy a new piece of machinery costing \$10,000. How much should the company deposit semiannually in a savings bank that pays 3% compounded semiannually in order to accumulate a fund * of \$10,000 at the end of 5 years?

13. A business concern plans to spend \$20,000 for an addition to its store 4 years from now. How much money should the concern deposit annually in a savings bank that pays 4% compounded annually in order to accumulate a fund * of \$20,000 at the end of four years?

14. A man deposits \$1000 at the end of each year for 8 years in a bank which pays 2%, compounded annually. During the next 4 years no pay-

* A fund created in this way to discharge an obligation due at some future date is called a **sinking fund.**

ments are made but the interest is compounded annually. What does the fund amount to at the end of the 12-year period?

15. A man buys a farm for \$15,000 and makes a down payment of \$3000. The remaining \$12,000 with 5% interest compounded annually is to be paid in 10 equal yearly installments.* How large is the annual payment?

16. A business concern owes a bank \$10,000. The bank agrees to let the firm discharge the debt in 5 equal annual payments,* the first at the end of a year. Interest is at 6%, compounded annually. What is the size of the annual payment?

17. A student borrows \$400 at the beginning of each of his four years in college. He agrees to pay the debt with 5% interest compounded annually in three equal annual payments, the first installment to be made five years after the first \$400 was borrowed. What is the annual payment?

18. Beginning on his 31st birthday, a man deposited \$500 in a bank each year for 35 years. Beginning on his 66th birthday, he withdrew \$1000 each year for 15 years. How much does he have to his account on his 80th birthday if the rate of interest is 3% compounded annually?

19. A man buys a house for \$7500. He pays \$1500 cash and promises to pay the remainder, including interest at 5% compounded semiannually, in 40 equal semiannual installments. How large is each payment?

20. The face value of a railroad bond is \$1000. Attached to the bond are 20 coupons. At the end of each year, for 20 years, the holder of the bond clips off a coupon and, through his banker, sends it to the railroad company which redeems it for \$25. At the end of 20 years the bond holder turns in the bond which is redeemed at par value, \$1000. If money is worth 3% compounded annually, find the present value of the bond and attached coupons.

* Discharging an obligation in this way is called **amortizing the debt.**

chapter 19 *Permutations and combinations*

161. Fundamental principle. Let R and S (Fig. 35) represent two stores on opposite sides of an arcade. Suppose that R has three doors A, B, C, and suppose that S has four doors, W, X, Y, Z. If a person leaves R by door A, he can enter S by any one of four different doors. Likewise if he leaves R by either of the other doors, B or C, he can then enter S in any one of four different ways. The $3 \cdot 4$ or 12 ways of going from R to S are: $AW, AX, AY, AZ, BW, BX, BY, BZ, CW, CX, CY, CZ$. This illustrates the fact that if one act can be performed in 3 ways and if a second act can be performed in 4 ways, then the two acts can be performed in the order stated, in $3 \cdot 4$ or 12 ways. In general we have the

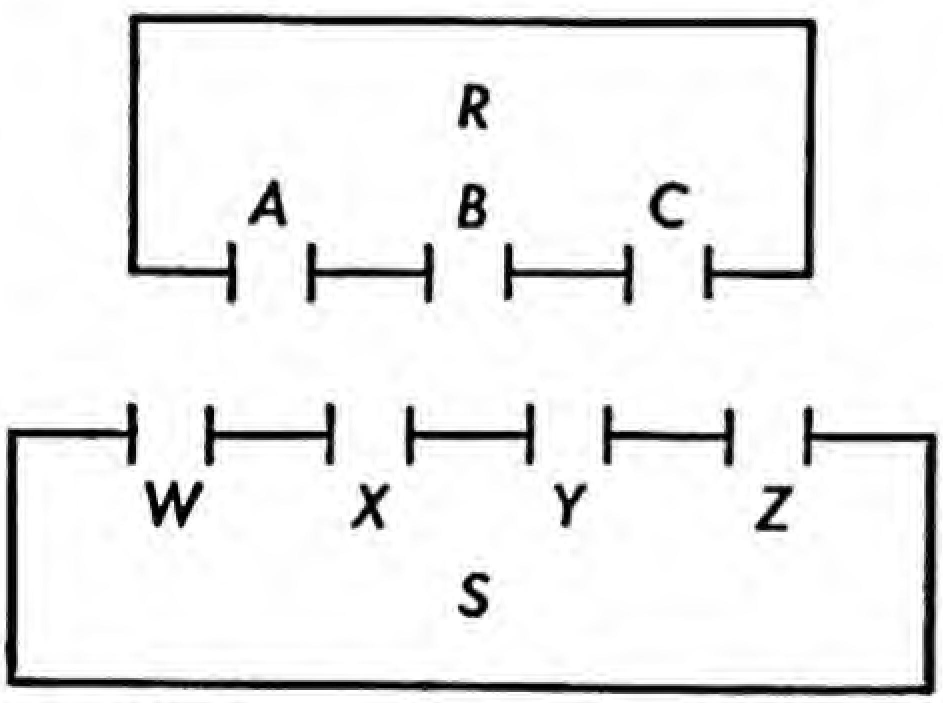


FIG. 35

*Fundamental Principle. If the first of two independent acts can be performed in **h** ways, and if the second act can be performed in **k** ways, then the number of ways of performing the two acts, in the order stated, is **hk**.*

This principle can be extended to three or more acts.

Example 1. Three people, A, B, C, get on a bus that has six vacant seats on each side. In how many ways can they be seated if A insists on sitting on the right side?

Solution. A can be seated in any one of 6 ways. After A sits down, B can take any one of the 11 remaining seats. After B has chosen his seat, C can select any one of the 10 remaining places. By the fundamental principle, the number of ways of seating the three people is $6 \cdot 11 \cdot 10 = 660$.

Exercise 84

1. A football stadium has 12 gates. In how many ways can a spectator enter by one gate and leave by another?

2. A man has 5 suits, 3 pairs of shoes, and 2 hats. How many different combinations of attire can he wear?

3. How many different balanced meals can be chosen from 2 kinds of soup, 6 meat courses, and 3 kinds of dessert?

4. In how many ways can a fraternity of 25 active members elect a president, a vice-president, and a treasurer if no person is to hold more than one office?

5. In how many ways can the eight teams in the Stony Mountain League finish the season if it is a foregone conclusion that Podunk Junction will be last and that the first two places will be filled by Allah Allah and Fresh Lake City in either order?

6. In a certain state, each auto license plate contains two letters (which can be the same) followed by a number from 1 to 9999. How many different plates can be made?

7. The Greek alphabet consists of 24 letters. How many names of fraternities can be formed by using 3 letters at a time, (a) if no repetitions are permitted, (b) if repetitions are permitted?

8. The alphabet of an artificial language has 12 letters. How many 3-letter words can be formed if no letter can occur more than once in the same word?

9. How many integers less than 7000, each containing four *different* digits, can be formed by using the digits 1, 5, 6, 8, 9?

10. How many *even* 3-digit numbers can be formed by using the digits 5, 6, 7, 8, 9, if no digit can occur more than once in the same number?

11. How many different batting orders can the captain of a baseball team present to the head umpire if the pitcher must bat last, and if one of the

three outfielders must bat first? The nine men who are to play have already been chosen.

12. Six people board a bus that has five vacant seats on each side. In how many ways can they be seated if two people insist on sitting on the right side and one person must sit on the left side?

13. A nickel, a dime, and a quarter are tossed onto a sidewalk. In how many ways can they fall? Actually write out these ways.

14. Each week during the football season, a case of soft drinks will be given to each person whose entry correctly predicts the winner (or a tie) in each of five listed games. How many different entries can be submitted?

15. If a regulation 52-card deck is used, how many 5-card poker hands with 4 aces can be dealt?

16. In how many ways can the letters of the word "certain" be arranged if the first and last places must be occupied by consonants?

162. Permutations and combinations. Let us consider four different things which we designate by the letters a, b, c, d. If we choose *groups of two* of these letters, we find the following 6 *combinations.*

ab	ac	ad	bc	bd	cd

If each of these combinations is arranged in *all possible orders,* we obtain the following 12 *permutations.*

ab	ac	ad	bc	bd	cd
ba	ca	da	cb	db	dc

Each different *arrangement* which can be made by taking all or a part of a set of things is called a **permutation** of the set.

163. Permutations of n different things taken r at a time. The symbol ${}_nP_r$ is read, "the number of permutations of n things taken r at a time." We have seen (Art. 162) that the number of permutations of 4 things taken 2 at a time is 12. Hence ${}_4P_2 = 12$.

We shall now derive a formula for ${}_nP_r$, where $n \geqq r$. Our problem is to find the number of ways of filling r places, using n things. We shall apply the fundamental principle. The first place can be filled in n ways. After it has been filled in any one of these ways, the second place can be filled in $(n - 1)$ ways. Hence the first two places can be filled in $n(n - 1)$ ways. The third place can be filled in $(n - 2)$

ways, $\cdots$, and the rth place can be filled in $(n - [r - 1])$ or $(n - r + 1)$ ways. Hence

$$_nP_r = n(n-1)(n-2)\cdots(n-r+1), \tag{1}$$

or

$$_nP_r = n(n-1)(n-2)\cdots \text{to } r \text{ factors.} \tag{2}$$

If numerator and denominator of the right side of (1) are multiplied by $(n-r)!$, we get

$$_nP_r = \frac{[n(n-1)(n-2)\cdots(n-r+1)](n-r)!}{(n-r)!},$$

or

$$_nP_r = \frac{n!}{(n-r)!}. \tag{3}$$

If $r = n$, we have $\quad {}_nP_n = \dfrac{n!}{0!}.$

Since 0! is defined (Art. 102) to be 1,

$$_nP_n = n! \tag{4}$$

Example 1. In how many ways can 3 marines and 4 soldiers be seated on a bench if the marines must be seated together?

Solution. We shall first consider the number of ways of choosing the *seats* to be occupied by marines. This can be done in 5 ways since the left-most marine has a choice of 5 seats (see diagram at right). Having chosen one of these 5 sets of seats for the marines, we shall next determine the number of ways to seat them. This is $_3P_3 = 3! = 6$. Finally, the soldiers, who must take the remaining 4 places, can be seated in $_4P_4 = 4! = 24$ ways. By the fundamental principle, the number of ways of seating the servicemen is $5 \cdot {}_3P_3 \cdot {}_4P_4 = 5 \cdot 6 \cdot 24 = 720$.

M	M	M	S	S	S	S
S	M	M	M	S	S	S
S	S	M	M	M	S	S
S	S	S	M	M	M	S
S	S	S	S	M	M	M

Example 2. In how many ways (i.e., arrangements relative to each other) can 5 people be seated at a round table?

Solution. First, place one person in any seat. Then arrange the remaining 4 people in all different ways *relative to the first person.*

This can be done in ${}_4P_4 = 4! = 24$ ways. Hence the number of ways the 5 people can be seated is 24. It is obvious that if, in a certain arrangement, each person moves to the seat at his right, the relative order remains the same.

The number of ways that n *things can be arranged in a circle is* $(n - 1)!$

164. Permutations of things some of which are alike. Let us consider the letters in the word "error," where the letter r occurs three times. If the letters were all different, e, r_1, r_2, o, r_3, there would be 5! permutations of them, including the following

$$\begin{array}{ccc} e\,r_1\,r_2\,o\,r_3 & e\,r_2\,r_1\,o\,r_3 & e\,r_3\,r_1\,o\,r_2 \\ e\,r_1\,r_3\,o\,r_2 & e\,r_2\,r_3\,o\,r_1 & e\,r_3\,r_2\,o\,r_1, \end{array}$$

which reduce to the same arrangement, $e\,r\,r\,o\,r$, if the subscripts are removed. Hence there are 3! or 6 times as many permutations when the r's are distinguishable as there are when the r's are identical. Therefore the number of permutations of the letters of the word "error" is $\dfrac{5!}{3!} = 20$.

A similar kind of reasoning gives us the following general result.

If a group of n *things consists of* n_1 *things of one kind,* n_2 *things of another kind,* n_3 *things of a third kind,* $\cdots$, *then the number* N *of distinct permutations of the* n *things taken* n *at a time is*

$$N = \frac{n!}{n_1!\,n_2!\,n_3!\cdots}.$$

Example 1. How many arrangements can be made of the letters of the word $T\,O\,R\,O\,N\,T\,O$?

Solution. The total number of letters is 7. The letter O occurs 3 times, and the letter T occurs 2 times. Hence

$$N = \frac{7!}{3!\,2!} = 420.$$

Exercise 85

1. Read and evaluate ${}_{12}P_3$.
2. Read and evaluate ${}_{100}P_2$.

3. How many permutations of two letters each can be formed from the letters a, b, c, d, e? Write out these permutations.

4. If seven horses run in the Kentucky Derby, in how many different orders can they finish?

5. In how many different orders can the eight teams in the National League finish the season?

6. In how many ways can 3 *different* prizes be awarded to 22 persons if no person is to receive more than one prize?

7. How many four-digit numbers can be formed from the numbers 1, 2, 6, 7, 8, 9, if no number can be repeated in any four-digit number?

8. Three people get on a bus that has ten vacant seats. In how many ways can they take their places?

9. In how many ways can 5 boys and 4 girls be seated on a bench if no two girls are to sit together, and the end seats must be occupied by boys?

10. In how many ways can 2 chemistry and 6 physics books be arranged on a shelf, (a) if the 2 chemistry books must be together, (b) if the 6 physics books must be together?

11. In how many ways can the letters of the word "arsenic" be arranged if the letter s must occupy either the first place or the last place?

12. In how many ways can the letters of the word "careful" be arranged if the consonants are to occupy the odd places?

13. In how many ways can 4 men and 4 women be seated at a round table if no two men are to be together?

14. In how many ways can 4 men and 3 women be seated at a round table if a certain man and his wife must occupy adjacent seats?

15. How many essentially distinct necklaces of 5 beads each can be formed by using 5 different beads?

16. In how many ways can 8 children join hands to form a circle?

17. In how many ways can the letters of each of the following words be arranged: (a) "entente," (b) "engineer"?

18. In how many ways can the letters of the word "mamma" be arranged? Write out these arrangements.

19. In how many ways can 3 nickels, 4 dimes, and 2 quarters be distributed to 9 children if each child is to receive one coin?

20. If ${}_nP_4 = {}_{n+1}P_3$, find n.

165. Combinations. Each group that can be formed by taking all or a part of a set of things, *without regard to the arrangement of the things in a group*, is called a **combination** of the set. The symbol ${}_nC_r$

is read "the number of combinations of n things taken r at a time." We have seen (Art. 162) that the number of combinations of 4 things taken 2 at a time is 6. Hence ${}_4C_2 = 6$.

We shall now derive a formula for ${}_nC_r$, where $n \geqq r$. To each combination of r things there will correspond $r!$ permutations of the same r things (equation 4 of Art. 163). This means that in choosing r things from n things, there are $r!$ times as many permutations as there are combinations. Hence ${}_nP_r = r!\,{}_nC_r$, or

$$ {}_nC_r = \frac{{}_nP_r}{r!} \tag{1} $$

Since

$$ {}_nP_r = \frac{n!}{(n-r)!}, $$

$$ {}_nC_r = \frac{n!}{r!\,(n-r)!} \tag{2} $$

Example 1. A lady has 12 friends. She wishes to invite 3 of them to a bridge party. How many times can she entertain without having the same group twice?

Solution. Since no reference has been made as to order or arrangement (e.g., the order in which the guests arrive or their seating arrangement at the bridge table), our problem is one of combinations rather than permutations. The number of groups is

$$ {}_{12}C_3 = \frac{12!}{3!\,9!} = \frac{\overset{10\cdot 11\cdot 12}{\cancel{12!}}}{3!\,\cancel{9!}} = \frac{10\cdot 11\cdot \overset{2}{\cancel{12}}}{1\cdot\cancel{2}\cdot\cancel{3}} = 220. $$

In evaluating ${}_{12}C_3$, notice that the top and bottom of the fraction are first divided by the *larger* factorial number in the denominator.

Example 2. From 10 men and 6 women, how many committees of 5 people can be chosen

(*a*) if each committee is to have exactly 3 men?
(*b*) if each committee is to have at least 3 men?

Solution. (*a*) The number of ways to choose 3 men from 10 men is ${}_{10}C_3 = 120$. The number of ways to choose 2 women from 6 women is ${}_6C_2 = 15$. By the fundamental principle, the number of ways of choosing the committee is ${}_{10}C_3 \cdot {}_6C_2 = 120 \cdot 15 = 1800$.

(*b*) If the committee is to contain *at least* 3 men, the possibilities are: 3 men and 2 women, 4 men and 1 woman, 5 men and no women.

The number of committees consisting of 3 men and 2 women is 1800. The number of committees containing 4 men and 1 woman is ${}_{10}C_4 \cdot {}_6C_1 = 210 \cdot 6 = 1260$.

The number of committees consisting of 5 men is ${}_{10}C_5 = 252$.

Hence the number of committees containing *at least* 3 men is

$$1800 + 1260 + 252 = 3312.$$

Notice that the formula for ${}_nC_r$ can be written in the form

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

which is the same as the coefficient of b^r in the binomial expansion (Art. 103, formula 2). Hence the binomial formula may be written

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_ra^{n-r}b^r + \cdots + {}_nC_nb^n.$$

166. Total number of combinations of *n* things. If, in the last equation, we set $a = b = 1$, we get

$$(1+1)^n = 1 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_n$$

or

$${}_nC_1 + {}_nC_2 + \cdots + {}_nC_n = 2^n - 1.$$

Hence the total number of combinations of n things, if they are taken 1, 2, 3, $\cdots$, or n at a time, is $2^n - 1$.

Exercise 86

1. Read and evaluate (*a*) ${}_{100}C_3$, (*b*) ${}_{100}C_{97}$.

2. How many combinations of three letters each can be formed from the letters a, b, c, d, e? Write out these combinations.

3. On a certain examination each student is to answer any 10 of the 12 questions. In how many ways can this choice be made?

4. In how many ways can a fraternity of 30 active members select 2 delegates to their national convention?

5. Twenty politicians meet at a party. How many handshakes are exchanged if each person shakes hands with each other person once and only once?

6. Prove that ${}_nC_r = {}_nC_{n-r}$.

7. In how many ways can 10 different books be divided equally between 2 boys?
Hint. Find the number of ways that the first boy can receive 5 books.

8. In how many ways can a class of 12 students be divided into two groups, one group to contain 7 while the other group is to consist of 5?

9. In how many ways can 9 different toys be distributed among children A, B, C, if A is to receive 4, B is to get 3, and C receives 2?

10. In how many ways can a committee of 4 seniors, 3 juniors, and 2 sophomores be chosen from 6 seniors, 8 juniors, and 10 sophomores?

11. From 20 Democrats and 12 Republicans, how many committees of 3 can be chosen if the Democrats are to have a majority on each committee?

12. The instructions on a certain examination are (*a*) answer 12 of the 15 questions, and (*b*) answer at least 4 of the first 5 questions. In how many ways can a student make his choice?

13. A boy has in his pocket a penny, a nickel, a dime, and a quarter. How many different sums of money can he take out if he removes one or more coins?

14. How many different committees of 3 men can be selected from 10 men if a certain two men refuse to serve together on any committee?

15. If six different coins are tossed, they can turn up in 64 ways. How many of these consist of 3 heads and 3 tails?

16. If four different coins are tossed, they can turn up in 16 ways. How many of these consist of (*a*) 4 heads, (*b*) 3 heads and 1 tail, (*c*) 2 heads and 2 tails, (*d*) 1 head and 3 tails, (*e*) 4 tails?

17. If a regulation 52-card deck is used, how many different 5-card poker hands can be dealt?

18. How many different 13-card bridge hands can be dealt?

19. How many different bridge hands consist of 10 spades, 2 hearts, and 1 club?

20. If a regulation 52-card deck is used, how many different 5-card poker hands will contain exactly 3 aces?

Exercise 87 (Miscellaneous Problems)

1. In a certain state each truck license plate contains 4 letters (repetitions are permitted). How many different plates can be formed?

2. How many triangles can be formed by using, as their vertices, the vertices of a regular polygon of 22 sides?

3. Each of the 8 teams of the American League plays 22 games with each other team. How many games are played?

4. In how many ways can 4 college beauties be chosen from 20 candidates?

5. Ten students drive daily from Slaton to Lubbock in 2 cars, 5 boys in each car. How many trips could they make before the same 5 boys ride together again?

6. In how many ways can 10 things be divided into two groups of 7 and 3?

7. A teacher has 6 different assignments for outside work to give to a group of 3 students. In how many ways can the assignments be distributed, two to each student?

8. Find n if ${}_nP_3 = 4 \cdot {}_{n+1}C_2$.

9. Given ${}_{24}C_r = {}_{24}C_{r-14}$, find ${}_{21}C_r$. *Hint.* See Ex. 86, problem 6.

10. Prove that ${}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$.

11. In how many different ways can the letters of the word "minimum" be arranged?

12. How many different signals can be made with 10 flags by placing them all at a time in a line on a flagpole if 2 flags are red, 3 are white, and 5 are blue?

13. In how many ways can 9 people be seated at a round table if a certain 3 people must occupy adjacent seats?

14. How many different bridge hands consist of 11 hearts and 2 black cards?

15. If a regulation 52-card deck is used, how many different 5-card poker hands consist of 4 hearts and 1 club?

16. If a regulation 52-card deck is used, how many different 5-card poker hands are flushes (all cards are of the same suit)?

17. The Greek alphabet consists of 24 letters. How many names of fraternities can be formed by using 3 letters at a time if at least 2 of the 3 letters must be the same?

18. How many 3-digit numbers can be formed by using the digits 1, 1, 6, 7, 8, 9?

19. In how many ways can one French book, two different German books, and four different Spanish books be placed in a line on a shelf if the Spanish books must be together and the German books must not be adjacent to each other?

20. How many different signals can be made with five different flags by placing one or more of them in a line on a flagpole?

chapter 20 *Probability*

167. Probability. When we say that a certain event is *possible*, we mean that it *can* happen. When we say that an event is *probable*, we imply that it is possible and we believe it is more likely to happen than not to happen.

If an event can happen in h *ways and fail to happen in* f *ways, and if all these ways are equally likely, then the probability of its happening is*

$$p = \frac{h}{h+f}$$

and the probability of the event failing to occur is

$$q = \frac{f}{h+f}.$$

This idea is also conveyed by the statements

(*a*) the odds are h to f in favor of the event, and

(*b*) the odds are f to h against the event.

For example, if one card is drawn from a deck,* the probability of getting a spade is $\frac{13}{13+39} = \frac{13}{52} = \frac{1}{4}$. This does not mean that out of every four draws exactly one spade will result. It does mean that, as the number of draws increases, the ratio of the number of spades to the number of draws will approach $\frac{1}{4}$. The odds are 1 to 3 in favor of drawing a spade and 3 to 1 against drawing a spade.

* In all card problems, it is understood that a regulation 52-card deck is used.

Example 1. A bag contains 10 red balls and 6 black balls. If 2 balls are drawn (at random), find the probability that

(*a*) Both are red.
(*b*) One is red and one is black.
(*c*) Both are black.

Solution. The total number of (equally likely) ways of drawing 2 balls from 16 balls is ${}_{16}C_2 = 120$.

(*a*) Two red balls can be drawn from 10 red balls in ${}_{10}C_2 = 45$ ways. Hence the probability of getting 2 red balls is

$$\frac{{}_{10}C_2}{{}_{16}C_2} = \frac{45}{120} = \frac{3}{8}.$$

(*b*) The number of ways of drawing a red ball and a black ball is ${}_{10}C_1 \cdot {}_6C_1 = 10 \cdot 6 = 60$. Therefore the probability of drawing one ball of each color is

$$\frac{{}_{10}C_1 \cdot {}_6C_1}{{}_{16}C_2} = \frac{60}{120} = \frac{1}{2}.$$

(*c*) The probability of drawing 2 black balls is

$$\frac{{}_6C_2}{{}_{16}C_2} = \frac{15}{120} = \frac{1}{8}.$$

Notice that the sum of the probabilities of the three possibilities is $\frac{3}{8} + \frac{1}{2} + \frac{1}{8} = 1$. A probability of 1 indicates certainty. One of these three events is bound to occur.

Example 2. In rolling four dice,* find the probability that at least one ace turns up.

Solution. Let p represent the probability of getting at least one ace, and let q be the probability of getting no ace. Since one of these two events must happen, $p + q = 1$.

We shall find q because this is much easier than finding p. The total number of (equally likely) ways that 4 dice can turn up is $6 \cdot 6 \cdot 6 \cdot 6 = 1296$. The number of ways the 4 dice can fall without

* In problems involving dice, it is understood that each die is a homogeneous cube with the numbers 1 (ace), 2, 3, 4, 5, 6, on its faces.

getting an ace is $5 \cdot 5 \cdot 5 \cdot 5 = 625$. Hence the probability of getting no ace is $\frac{5^4}{6^4} = \frac{625}{1296}$. Therefore the probability of getting at least one ace is

$$p = 1 - q = 1 - \tfrac{625}{1296} = \tfrac{671}{1296},$$

or approximately 52%.

168. Relative frequency. Empirical probability. Sometimes it is impossible to determine, by logical analysis, the number of equally likely ways in which an event can happen or fail to happen. We then resort to experiment or observation. If in a series of n trials of an event, the event has been observed to happen h times, then $\frac{h}{n}$ is called the **relative frequency** of occurrence of the event. If n is a large number, we use the relative frequency as an approximation for probability. This type of probability is said to be **empirical** (determined by experiment) as in contrast to mathematical probability.

Example 1. In a school with an enrollment of 5000 students, 1800 students were selected at random and it was found that 150 of them were left-handed. What is the probability that a student chosen at random is left-handed?

Solution. The relative frequency of a student being left-handed is $\frac{150}{1800} = \frac{1}{12}$. Since 1800 represents a fairly good sampling, the

$$\text{empirical probability} = \tfrac{1}{12}, \text{ approximately.}$$

Comment. If the sampling of 1800 students represented the *total* enrollment of the school, then the

$$\text{mathematical probability} = \tfrac{1}{12}, \text{ exactly.}$$

A good illustration of the use of empirical probability is provided in the American Experience Table of Mortality (Table VIII). This table is used by life insurance companies in calculating the proper premiums for people of various ages.

Example 2. Find the probability that a person aged 18 will live to be 65.

Solution. Table VIII shows that of 94,089 people alive at age 18, only 49,341 will still be alive at age 65. Hence the probability is $\frac{49341}{94089} = .5244$.

169. Expectation. If a person is to receive a certain sum of money M in case a given event occurs, and if the probability of the event's happening is p, then the value of the person's **expectation** is pM.

Illustration. If a person is to receive \$10 in case he draws a spade from a deck of cards, then the value of his expectation is $\frac{1}{4}(\$10) = \$2.50$. This represents the amount of money he should pay for the privilege of making the draw.

Exercise 88

1. The odds are 8 to 5 against an event's happening. What is the probability that the event will occur?

2. What is the probability that a boy born in 1947 has his birthday in October?

3. In a certain lottery, 200 tickets are sold. If a single prize of \$60 is offered, find the expectation of a person who has 5 tickets.

4. A person is to receive \$3 if he throws an ace with a die. (*a*) Find his expectation. (*b*) What are the odds against his getting an ace?

5. An urn contains six black, four red, and two white balls. If two balls are drawn, find the probability that
(*a*) Both are black.
(b) One is red and one is white.
(*c*) Neither is white.

6. A bag contains seven black and three red balls. If three balls are drawn, find the probability that
(*a*) All are black.
(*b*) Two are black and one is red.
(*c*) At least one is black.

7. In drawing two cards from a deck, find the probability of getting
(*a*) Two clubs.
(*b*) One club and one heart.
(*c*) "Black jack" (an ace and one of the following: king, queen, jack, or ten).

8. In drawing two cards from a deck, find the probability of getting
(*a*) An ace and a king.
(*b*) No red card.

(*c*) A total of 8 points, if an ace counts one point, a face card counts 10 points, and all other cards are counted by their face value.

9. A poker player holds four clubs and a heart. He discards the heart and draws one card. What is the probability that he will get another club?

10. A poker player holds the following five cards: 4, 5, 6, 7, 10. He discards the 10 and draws one card. What is the probability that he will get a 3 or an 8?

11. An ordinary coin is tossed 50 times with the result: 40 heads and 10 tails. If the tosses are continued until a total of 100 have been made, what is the most probable number of tails in the second 50 tosses?

12. Let each member of the class toss a coin 20 times and record the number of times a head turns up. Total the results for the class and find the relative frequency of getting a head.

Use the American Experience Table of Mortality to find the specified probability. Leave the result in fractional form.

13. That a person aged 19 will live to be 50.

14. That a person aged 70 will die within one year.

15. That a person aged 20 will die within 5 years.

16. That a person aged 50 will live at least 10 more years.

17. In which year of life will a person now 18 years of age be most likely to die?

18. Four French and five Spanish books are placed at random on a shelf. What is the probability that the French and Spanish books alternate?

19. If two of the integers from 1 to 9 inclusive are selected at random, find the probability that (*a*) both are odd, (*b*) at least one is odd.

20. If two coins are tossed, find the probability that (*a*) both are heads, (*b*) one is a head and the other is a tail, (*c*) both are tails. Find the sum of these three probabilities.

21. In throwing two dice, find the probability of getting a total of 8.
Hint. The following equally likely groupings produce a total of 8: 2–6, 3–5, 4–4, 5–3, 6–2.

22. If two dice are thrown, find the probability of getting a 10 before a 7 turns up.

170. Mutually exclusive events. Two events are said to be **mutually exclusive** if not more than one of them can happen in the same trial. For example, the drawing of a spade and the drawing

of a heart from a deck of cards are mutually exclusive events. Drawing a spade and drawing a king are not mutually exclusive events.

Theorem. If two mutually exclusive events have the separate probabilities p_1 and p_2, then the probability that either the first or the second event will happen is $p_1 + p_2$.

Proof. Let n be the total number of (equally likely) ways the events can happen or fail to happen. Let h_1 be the number of ways the first event can happen and let h_2 be the number of ways the second event can happen. Then, by the definition of probability, $p_1 = \frac{h_1}{n}$ and $p_2 = \frac{h_2}{n}$. Since the two events are mutually exclusive, the number of ways that one or the other can happen is $h_1 + h_2$. Hence the probability that one or the other of the two events will happen is

$$\frac{h_1 + h_2}{n} = \frac{h_1}{n} + \frac{h_2}{n} = p_1 + p_2.$$

The theorem and its proof can be readily extended to the case of more than two mutually exclusive events.

Illustration 1. A piggy bank contains 2 quarters, 4 dimes and 9 pennies. If a coin is shaken out, the probability that it is a quarter is $\frac{2}{15}$, and that it is a dime is $\frac{4}{15}$. Hence if one coin is removed, the probability that it is a silver coin (i.e., either a quarter or a dime) is $\frac{2}{15} + \frac{4}{15} = \frac{6}{15} = \frac{2}{5}$.

171. Independent events. Two or more events are said to be **independent** if the occurrence of one of them does not affect the probability of occurrence of the others. For example, drawing an ace from a deck of cards and getting heads when a coin is tossed are two independent events.

Theorem. The probability that all of a set of independent events will happen is the product of their separate probabilities.

Proof. Consider two independent events E_1 and E_2. Suppose that E_1 happens h_1 times out of n_1 trials. Then the probability p_1 that

E_1 will happen is $p_1 = \frac{h_1}{n_1}$. Suppose that E_2 happens h_2 times out of n_2 trials. Then the probability p_2 that E_2 will happen is $p_2 = \frac{h_2}{n_2}$. By the fundamental principle (Art. 161), *both* E_1 and E_2 will happen h_1h_2 times out of n_1n_2 trials. Hence the probability that both events will happen is

$$\frac{h_1h_2}{n_1n_2} = \frac{h_1}{n_1} \cdot \frac{h_2}{n_2} = p_1p_2.$$

The proof can be readily extended to the case of more than two independent events.

Illustration 1. If the probability of a man's winning his golf game is $\frac{1}{3}$, and if the probability of his wife's winning her bridge game is $\frac{1}{4}$, then the probability that they both win is $\frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$.

The probability that they both lose is $(1 - \frac{1}{3})(1 - \frac{1}{4}) = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$.

The probability that he wins and she loses is $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$.

The probability that he loses and she wins is $\frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}$.

Notice that the sum of the four probabilities is 1:

$$\tfrac{1}{12} + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{6} = 1.$$

172. Dependent events. If the occurrence of a first event affects the probability of occurrence of a second event, then the second event is said to be **dependent** upon the first.

Theorem. If the probability of occurrence of a first event is p_1 and if, after this has happened, the probability of occurrence of a second event is p_2, then the probability that both events will happen in the order stated is p_1p_2.

The proof is similar to that for the case of independent events.

The theorem can be extended to the case of more than two dependent events.

Example 1. Two cards are drawn from a deck. If the first card is not replaced before the second is drawn, find the probability that both are spades.

Solution. The probability of getting a spade on the first draw is $\frac{13}{52} = \frac{1}{4}$. If the first card is a spade, then the remaining 51 cards will

contain only 12 spades. Hence the probability of drawing a second spade (after getting a spade on the first draw) is $\frac{12}{51} = \frac{4}{17}$. Therefore the probability of getting two spades is $\frac{1}{4} \cdot \frac{4}{17} = \frac{1}{17}$.

Comment 1. If the two cards are drawn simultaneously (rather than successively), the probability of getting two spades is also $\frac{1}{17}$. This can be verified by the method of Ex. 1, Art. 167:

$$\text{Probability} = \frac{{}_{13}C_2}{{}_{52}C_2} = \frac{13 \cdot 6}{26 \cdot 51} = \frac{1}{17}.$$

Comment 2. If the first card is replaced before the second is drawn, the probability of getting two spades is $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$.

Example 2. If three dice are thrown, find the probability that (*a*) all will be different, (*b*) only two will be alike, (*c*) all three will be alike.

Solution. (*a*) Let the first die fall in any one of the six possible ways. Then the probability that the second die will be different from the first is $\frac{5}{6}$. The first two dice having fallen in different ways, the probability that the third die is different from the first two is $\frac{4}{6}$. Hence the probability that all three dice are different is $\frac{5}{6} \cdot \frac{4}{6} = \frac{5}{9}$.

(*b*) Using an argument similar to that in (*a*), we see that the probability that the *first two* dice are alike and the *third* one is different is $\frac{1}{6} \cdot \frac{5}{6} = \frac{5}{36}$. But the two dice that are alike could be the first and second, or the first and third, or the second and third. Hence the probability that *any two* dice will be alike and the other one will be different is $3(\frac{1}{6})(\frac{5}{6}) = \frac{5}{12}$.

(*c*) The probability that all three will be alike is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

Notice that the sum of the three probabilities is 1.

Exercise 89

1. Three brothers, Tom, Dick, and Harry, enter the same swimming meet. The probability that the race will be won by Tom is $\frac{1}{3}$, by Dick $\frac{1}{4}$, by Harry $\frac{1}{6}$. Find the probability that one of the brothers will win the race.

2. The probability that a boy wins his tennis match is $\frac{1}{3}$. The probability that his father wins his golf game is $\frac{5}{6}$. The probability that his grandfather wins his checkers game is $\frac{1}{2}$. What is the probability that they all lose?

3. If a coin is tossed three times, what is the probability that it will fall heads each time?

4. A coin purse contains three pennies, four nickels, and five dimes. If one coin is drawn at random, what is the probability that it is a penny or a nickel?

5. The probability that a certain man will live 20 years is $\frac{3}{5}$, and that his wife will live 20 years is $\frac{5}{8}$. Find the probability that (*a*) both man and wife will live 20 years, (*b*) the man will live and his wife will not live 20 years, (*c*) one of them will live and the other will not live 20 years.

6. A person draws a card from each of two different decks. Find the probability that (*a*) both are spades, (*b*) the first card drawn is an ace and the second one is a spade, (*c*) one card is an ace and the other is a spade.

7. In rolling two dice, find the probability of rolling a 7 or an 11 on the first try.

8. The odds * in favor of the New York Yankees' winning the next American League pennant are 3 to 2. The odds * in favor of their winning in World Series competition are 5 to 1. What is the probability that the Yankees will be the next World's Champions?

9. The odds * against Detroit's winning the American League pennant are 4 to 1. The odds * against St. Louis in the National League are 3 to 2. What is the probability that they will meet in the next World Series?

10. Two hundred tickets are sold in a lottery which awards two prizes. If I buy five tickets, what is the probability that I will win something?

11. Adams and Barton are hunting ducks. Adams averages 9 hits out of 10 shots while Barton averages 7 hits out of 10 shots. What is the probability that they will get a duck at which both are shooting?

12. Three cards are drawn in succession from a regulation deck. Find the probability that they will be of different suits.

173. Repeated trials. *Theorem. If p is the probability that an event will happen and q is the probability that it will fail in a single trial, then the probability that the event will happen exactly r times out of n trials is*

$$_nC_r p^r q^{n-r}.$$

Proof. The probability that the event will happen each time in r *specified* trials (such as the first r trials), and fail to happen in the remaining $(n - r)$ trials is $p^r q^{n-r}$, by Art. 171. But these r trials can be selected from n trials in $_nC_r$ ways, which are mutually exclusive. Using Art. 170, we find that the probability in question is

$$p^r q^{n-r} + p^r q^{n-r} + \cdots \text{ to } {_nC_r} \text{ terms} = {_nC_r} p^r q^{n-r}.$$

* Based on their performances in the 20-year period, 1926–1945.

Corollary. The probability that an event will happen at least r *times out of* n *trials is*

$$p^n + {}_nC_{n-1}p^{n-1}q + {}_nC_{n-2}p^{n-2}q^2 + \cdots + {}_nC_rp^rq^{n-r}.$$

This follows from the fact that the terms of this expression represent, respectively, the probabilities of the event happening exactly n times, exactly $(n - 1)$ times, exactly $(n - 2)$ times, $\cdots$, exactly r times. Notice that the expression is the sum of the first $(n - r + 1)$ terms of the expansion of $(p + q)^n$.

Example 1. If a die is thrown 5 times, find the probability of getting (*a*) exactly 3 aces, (*b*) at least 3 aces, (*c*) at least 1 ace.

Solution. (*a*) The probability of throwing an ace in a single trial is $\frac{1}{6}$. The probability of not throwing an ace is $\frac{5}{6}$. Hence the probability of getting an ace on each of the first three throws but not on the next two throws is

$$\tfrac{1}{6}\cdot\tfrac{1}{6}\cdot\tfrac{1}{6}\cdot\tfrac{5}{6}\cdot\tfrac{5}{6} = (\tfrac{1}{6})^3(\tfrac{5}{6})^2.$$

But the 3 aces can appear on any 3 of the 5 throws, i.e., they can occur in ${}_5C_3 = \dfrac{5!}{3!\,2!} = 10$ ways.* The probability of getting 3 aces in *any* specified *one* of these 10 orders is $(\frac{1}{6})^3(\frac{5}{6})^2$. Hence the probability of getting 3 aces in any order is

$$ {}_5C_3(\tfrac{1}{6})^3(\tfrac{5}{6})^2 = \tfrac{125}{3888}.$$

(*b*) The probability of getting *at least* 3 aces is

$$\begin{aligned} &p^5 + {}_5C_4p^4q + {}_5C_3p^3q^2 \\ &= (\tfrac{1}{6})^5 + 5(\tfrac{1}{6})^4(\tfrac{5}{6}) + 10(\tfrac{1}{6})^3(\tfrac{5}{6})^2 \\ &= \frac{1}{6^5} + \frac{25}{6^5} + \frac{250}{6^5} = \frac{276}{7776} = \frac{23}{648}. \end{aligned}$$

(*c*) The probability that at least one ace is thrown can be found most easily by (1) finding the probability that no ace is thrown, and then (2) subtracting this result from 1.

* If the occurrence of an ace is indicated by A and the non-occurrence by x, then these 10 ways are

$AAAxx$	$AAxAx$	$AAxxA$	$AxAAx$	$AxAxA$
$AxxAA$	$xAAAx$	$xAAxA$	$xAxAA$	$xxAAA$

The probability of getting no ace is $(\frac{5}{6})^5$. Hence the probability of getting at least one ace is

$$1 - (\tfrac{5}{6})^5 = 1 - \tfrac{3125}{7776} = \tfrac{4651}{7776}.$$

Exercise 90

1. The probability that a certain basketball player will "make" a free throw is $\frac{2}{3}$. If he attempts 4 free throws, find the probability that he (*a*) makes 4, (*b*) makes 3, (*c*) makes 2, (*d*) makes 1, (*e*) misses all 4.

2. A certain baseball player's batting average is .300.* Find the probability of his getting (*a*) exactly 3 hits in 4 times at bat, (*b*) at least 2 hits in 4 times at bat.

3. The probability that A will defeat B in a set of tennis is $\frac{3}{5}$. Find the probability that A will win at least 2 sets out of 3 from B.

4. The probability that a certain golfer will shoot par on a given hole is $\frac{4}{5}$. If he plays the hole 3 times, find the probability that he (*a*) pars it 3 times, (*b*) pars it only twice, (*c*) pars it only once, (*d*) does not par it at all.

5. A die having the shape of a regular tetrahedron has its faces numbered 1, 2, 3, 4. If the die is thrown five times, find the probability that an ace turns up (*a*) exactly two times, (*b*) at least two times.

6. A coin is tossed 6 times. Find the probability of getting (*a*) exactly 4 heads, (*b*) at least 4 heads.

7. Weather statistics indicate that a certain community receives rain on 73 days out of an ordinary 365-day year. Find the probability that (*a*) no rain will fall next monday, (*b*) rain will fall tomorrow and also the next day, (*c*) there will be rain on next Monday but not on Tuesday or Wednesday, (*d*) in the next three days, there will be rain on one day but not on the other two days.

8. If a die is thrown 7 times, find the probability of getting (*a*) exactly 2 aces, (*b*) exactly 6 aces, (*c*) at most 6 aces.

Exercise 91 (Miscellaneous Problems)

1. What is the probability that a person's birthday falls on Feb. 29? (Assume that *every* fourth year is a leap year.)

2. Find the odds against an event if its probability of happening is $\frac{3}{8}$.

* This means that he has averaged 3 hits in 10 times at bat. The probability of his getting a hit in one time at bat is $\frac{3}{10}$.

3. If two cards are drawn from a deck, find the probability that (*a*) the first is an ace and the second is a queen, (*b*) one card is an ace and the other is a queen, (*c*) at least one card is an ace.

4. The probability that a certain student can solve a given problem is $\frac{2}{3}$. The probability that his roommate can solve the problem is $\frac{1}{3}$. Find the probability that one or both of them can solve it.

5. A coin is tossed 5 times. Find the probability that the first 2 tosses result in heads, while at least 2 of the last 3 tosses result in tails.

6. A bag contains 2 black, 3 red, and 5 white balls. If 3 balls are drawn, find the probability of getting (*a*) one ball of each color, (*b*) 2 red balls and 1 white ball, (*c*) 3 black balls.

7. Six people are seated at random at a round table. Find the probability that a certain two people have adjacent seats.

8. If 2 algebra books and 4 trigonometry books are placed at random on a shelf, what is the probability that the algebra books are together?

9. A punch-board contains one \$5 prize, ten \$1 prizes and 489 blanks. Find the expectation of a person who buys one punch.

10. An envelope contains two \$5 bills and eight \$1 bills. Find the expectation of a person who draws one bill at random.

11. The probability that a marksman will hit the bull's-eye is $\frac{4}{5}$. Find the probability that he will hit it exactly 4 times out of 5 tries.

12. If q is the probability that an event will fail to happen in a single trial, prove that the probability that it will happen at least once in n trials is $(1 - q^n)$.

One box contains 4 red balls and 8 black balls. Another box contains 3 red balls and 7 black balls. Find the probability of the specified event.

13. If one ball is drawn from each box, (*a*) both will be red, (*b*) both will be of the same color.

14. If a ball is drawn from a box selected at random, it will be black.

15. If two balls are drawn from each box, all will be red.

16. From a group of 6 men and 4 women, a committee of 4 is chosen by lot. Find the probability that the committee will contain (*a*) 4 men, (*b*) 4 people of the same sex, (*c*) 3 men and 1 woman, (*d*) at least 3 men.

17. One humidor contains 12 Lucky Strikes and 8 Camels. A second humidor contains 6 Lucky Strikes and 10 Camels. If a person selects a humidor at random and then takes a cigarette, what is the probability that he draws a Lucky Strike?

18. A die is thrown 6 times. Which is more probable: (*a*) getting at least 2 aces, or (*b*) getting exactly 1 ace?

19. Four cards are drawn from a deck. Find the probability of getting a club, a diamond, a heart, and a spade (*a*) in that order, (*b*) in any order.

20. Show that the probability of winning when "shooting craps" * with two dice is $\frac{244}{495}$.

* For the benefit of those who are not familiar with this "he-stoops-to-conquer" pastime: (1) the person rolling the dice wins immediately if he gets 7 or 11; (2) he loses immediately if he gets 2, 3, or 12; (3) if he rolls any other point, he wins if he gets this point again *before* he rolls a 7.

chapter 21 *Determinants*

174. Introduction. In Arts. 43 and 44 we defined determinants of orders 2 and 3:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

We shall now define a determinant of order n in such a way as to include the foregoing definitions as special cases.

175. Inversions. In order to simplify the definition of a determinant of order greater than 3, we introduce the idea of inversion. Let us consider various permutations of the first n positive integers. We say that an **inversion** occurs whenever a number precedes a smaller number. For example, the permutation 4231 contains the five following inversions: 4 precedes 2; 4 precedes 3; 4 precedes 1; 2 precedes 1; and 3 precedes 1.

176. Determinants of any order. A determinant of order n is defined to be a square array of n^2 numbers called **elements,** arranged in n rows and n columns and enclosed by two vertical bars:

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \cdots l_1 \\ a_2 & b_2 & c_2 \cdots l_2 \\ a_3 & b_3 & c_3 \cdots l_3 \\ . & . & . \quad . \quad . \\ a_n & b_n & c_n \cdots l_n \end{vmatrix}.$$

The value of this determinant is defined to be the algebraic sum of all possible products of n *factors which can be formed by*

(1) *choosing one and only one element from each row and each column of* D *and*

(2) *placing before each such product a* **plus** *or* **minus** *sign according as the number of inversions of the subscripts is* **even** * *or* **odd** *when the letters have been written in the same order as in the first row of the determinant.*

The student should apply this general definition to the third-order determinant given in Art. 174 and verify that it is equivalent to the expansion listed there.

Corollary. *The expansion of a determinant of order* n *consists of* $n!$ *terms.*

Proof. The number of terms is equal to the number of permutations of the subscripts 1, 2, 3, $\cdots$, n. This is ${}_nP_n = n!$

177. Properties of determinants.

Property 1. *The value of a determinant is unchanged if corresponding rows and columns are interchanged;* i.e., the first row becomes the first column, the second row the second column, etc.

Illustration 1.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \text{let } D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Property 1 says that $D = D'$. The student should verify this by expanding both determinants by the method of Art. 44.

Proof. Let D represent any general nth order determinant and let D' be the determinant obtained by interchanging rows and columns. Then, choosing one element from each row and column, we see that the various products obtained from D are the same as those from D'. This means that the terms of the expansion of D are the same, except possibly for sign, as those of D'. The signs are identical because the number of inversions of subscripts when the

* Zero is considered as an even number.

letters are in natural order is equal to the number of inversions of letters when the subscripts are in natural order. For example, in Illustration 1, in the expansion of D, the term $a_2b_3c_1$ has two inversions of *subscripts* while the corresponding term in the expansion of D', $b_1c_2a_3$, has two inversions of *letters*.

From property 1 we conclude that for every theorem concerning the columns of a determinant there is a corresponding theorem concerning the rows, and vice versa.

Property 2. *If all the elements of any column (or row) are equal to zero, then the determinant is equal to zero.*

Proof. Each term of the expansion contains one factor from this column (or row) of zeros. Hence each term of the expansion is zero.

Property 3. *If two columns (or rows) are interchanged, the sign of the determinant is changed.*

Illustration 2.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and} \quad \text{let } D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Property 3 says that $D' = -D$. The student should verify this by expanding the two determinants and noticing that each term of D' is the negative of a term of D.

Proof. Let D represent any general nth order determinant.

Case 1. Suppose that two *adjacent* rows of D are interchanged. This produces an interchange of two adjacent subscripts in each term of the expansion. Therefore the number of inversions will be increased by one or decreased by one in each term. Hence each term will be changed in sign and the determinant will be changed in sign.

Case 2. Suppose that we interchange two rows that are separated by k intermediate rows. Moving the lower row up to a position immediately below the upper one requires k interchanges of adjacent rows. Moving the upper row down to the position originally occupied by the lower row requires $(k + 1)$ interchanges of adjacent rows. Hence the total number of interchanges is $(2k + 1)$, which is

an odd number for all positive integral values of k. Therefore the interchange of *any* two rows produces an odd number of changes of sign, i.e., the sign of the determinant is changed.

Property 4. *If two columns (or rows) of a determinant are identical, the value of the determinant is zero.*

Illustration 3. Property 4 says that

$$\begin{vmatrix} a & r & a \\ b & s & b \\ c & t & c \end{vmatrix} = 0.$$

Proof. Let the value of the determinant be D. By Property 3, if the two identical columns are interchanged, the value of the new determinant is $-D$. But the interchange of two *identical* columns leaves us with the original determinant D. Hence $D = -D$, $2D = 0$, or $D = 0$.

Property 5. *If each of the elements of a column (or row) of a determinant is multiplied by the same number* k, *then the value of the determinant is multiplied by* k.

Illustration 4.

$$\begin{vmatrix} a & b & kc \\ d & e & kf \\ g & h & ki \end{vmatrix} = k\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

Proof. Each term in the expansion of the new determinant will contain one and only one element from this column. Hence each term will contain one and only one factor k, i.e., it will be k times the corresponding term of the original determinant.

Property 6. *If each element of some column (or row) is expressed as the sum of two numbers, then the determinant may be expressed as the sum of two determinants.*

Illustration 5.

$$\begin{vmatrix} a & b & c+r \\ d & e & f+s \\ g & h & i+t \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & r \\ d & e & s \\ g & h & t \end{vmatrix}.$$

Proof. Each term of the expansion of the original determinant is equal to the sum of the corresponding terms of the other two determinants. For example, in Illustration 5, $ae(i + t) = aei + aet$.

Property 7. *The value of a determinant is not changed if to each element of any column (or row) we add* k *times the corresponding element of another column (or row).*

Proof (for a third-order determinant).

$$\text{Let } D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad \text{and} \quad \text{let } D' = \begin{vmatrix} a & b & c + ka \\ d & e & f + kd \\ g & h & i + kg \end{vmatrix}.$$

In order to show that $D' = D$, apply Property 6 to D':

$$\begin{aligned} D' &= \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} a & b & ka \\ d & e & kd \\ g & h & kg \end{vmatrix} \\ &= D + k \begin{vmatrix} a & b & a \\ d & e & d \\ g & h & g \end{vmatrix} \\ &= D + k \cdot 0 = D. \end{aligned}$$

178. Minor of an element. The **minor** of an element is the determinant that remains after we strike out the row and the column in which the element appears. For example, in the determinant

$$\begin{vmatrix} 3 & 4 & 5 \\ 16 & 17 & 18 \\ x & y & z \end{vmatrix}, \quad \text{the minor of } y \text{ is} \quad \begin{vmatrix} 3 & 5 \\ 16 & 18 \end{vmatrix}.$$

179. Expansion of a determinant by minors. *Theorem. The value of a determinant of order* n *may be found by expanding it by minors according to a given column (or row):*

1. *Multiply each element of the column by its minor and assign to each product a* **plus** *or a* **minus** *sign according as the sum of the number of the row and the number of the column containing the element is* **even** *or* **odd.**

2. *The sum of these* n *signed products is equal to the value of the determinant.*

Illustration 1. The following determinant is expanded by minors according to the first row.

$$\begin{vmatrix} a & b & c \\ 7 & 8 & 9 \\ 16 & 17 & 18 \end{vmatrix} = a(+)\begin{vmatrix} 8 & 9 \\ 17 & 18 \end{vmatrix} + b(-)\begin{vmatrix} 7 & 9 \\ 16 & 18 \end{vmatrix} + c(+)\begin{vmatrix} 7 & 8 \\ 16 & 17 \end{vmatrix}.$$

To the first determinant we assign a $+$ sign because the element a is in row 1 and column 1; $1 + 1 = 2$, an even number; hence the sign is $+$. To the second determinant we assign a $-$ sign because the element b is in row 1 and column 2; $1 + 2 = 3$, an odd number; hence the sign is $-$.

Illustration 2. The following determinant is expanded by minors according to the 2nd column.

$$\begin{vmatrix} 6 & 15 & 10 & 2 \\ 7 & 16 & 11 & 3 \\ 8 & -17 & -20 & 4 \\ 9 & 0 & -30 & 5 \end{vmatrix} = (15)(-)\begin{vmatrix} 7 & 11 & 3 \\ 8 & -20 & 4 \\ 9 & -30 & 5 \end{vmatrix} + (16)(+)\begin{vmatrix} 6 & 10 & 2 \\ 8 & -20 & 4 \\ 9 & -30 & 5 \end{vmatrix}$$

$$+ (-17)(-)\begin{vmatrix} 6 & 10 & 2 \\ 7 & 11 & 3 \\ 9 & -30 & 5 \end{vmatrix} + (0)(+)\begin{vmatrix} 6 & 10 & 2 \\ 7 & 11 & 3 \\ 8 & -20 & 4 \end{vmatrix}.$$

Hence we see that the value of the given fourth-order determinant can be obtained by expanding three third-order determinants. (The fourth one need not be evaluated since it is multiplied by zero.)

Note. If we are expanding a determinant by minors according to a certain column (or row), it is desirable to have as many zeros as possible in this column (or row). A method of obtaining zeros by using property 7 is explained in Art. 180.

Proof of the theorem. We must show that the expansion of the determinant by minors according to any column produces exactly the same terms that we obtain if we apply the definition of the value of a determinant. To do this, we shall establish the following facts.

(I) If a_1 is the element in the upper left-hand corner of the determinant D, and if A_1 is the minor of a_1, then, in the expansion of D, all terms involving a_1 are given by a_1A_1.

Each term involving a_1 is obtained by multiplying a_1 by one and only one element of each of the remaining rows and columns, i.e., by a term of its minor A_1. Furthermore, in the expansion of D, each term involving a_1 will have the same sign as the corresponding term formed by multiplying a_1 by the proper term of A_1 because writing a_1 in front of the various terms of A_1 will not change the number of inversions in any of these terms.

(II) If e is the element in the ith row and jth column of D, and if E is the minor of e, then, in the expansion of D, all terms involving e are given by eE or $-eE$ according as $(i + j)$ is even or odd.

The element e can be moved from the ith row to the first row by $(i - 1)$ interchanges of adjacent rows. This places the element e in the first row and jth column. Then e can be moved from the jth column to the first column by $(j - 1)$ interchanges of adjacent columns. Hence the element e can be moved to the upper left-hand corner of the determinant by $(i - 1) + (j - 1) = (i + j - 2)$ interchanges of rows and columns.

Let D' be the new determinant formed when e assumes the position previously occupied by a_1. Then, by property 3,

$$D' = (-1)^{i+j-2}D = (-1)^{i+j}D.$$

These interchanges of rows and columns will not affect the relative positions of the elements that lie outside of the ith row and jth column. Hence the minor of e remains the same.

By (I), the terms of D' involving e are given by eE. Hence the terms of D involving e are $(-1)^{i+j}eE$. This means that the terms involving e are given by eE if $(i + j)$ is even, and by $-eE$ if $(i + j)$ is odd.

The proof of the theorem is complete if we recall that each term of the expansion of D involves one and only one element from the given column. Hence we may write the expansion of D by minors according to this column.

Corollary. If, in the expansion of a determinant by minors according to a given column (or row), the elements of this column (or row) are replaced by the corresponding elements of another column (or row), then the resulting expansion is equal to zero.

Proof. The preceding theorem enables us to write

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4$$

where A_1 is the minor of a_1, A_2 is the minor of a_2, etc. If the a's on the right side of this equation are replaced by the corresponding b's, we get

$$b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4.$$

To show that this expression is equal to zero, notice that it can be considered as the expansion of the following determinant by minors according to the first column:

$$\begin{vmatrix} b_1 & b_1 & c_1 & d_1 \\ b_2 & b_2 & c_2 & d_2 \\ b_3 & b_3 & c_3 & d_3 \\ b_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

By property 4, this determinant is equal to zero.

180. Evaluating a determinant of order greater than two. A seoond-order determinant should be evaluated by the method of Art. 43. For determinants of order greater than 2, the following procedure is usually desirable:

1. *Choose a certain column (or row) and use property 7 to reduce all elements, except one, of this column (or row) to zeros.*

2. *Expand the determinant by minors according to the column (or row) that contains the zeros.* This reduces the original determinant to a new determinant whose order is one less.

3. *Repeat Steps 1 and 2 until a determinant of order two is obtained.*

Example 1. Evaluate $\begin{vmatrix} 6 & 0 & 8 & 5 \\ -6 & -2 & -5 & 8 \\ 5 & 1 & 4 & -6 \\ 22 & 3 & 19 & -18 \end{vmatrix}.$

Solution. The only zero element appears in the 1st row and 2nd column. None of the remaining elements in the 1st row is 1 or -1.

In the 2nd column, however, we do find the element 1. We shall try to get zeros in the 2nd column.*

We shall use property 7 to replace the elements -2 and 3 in the *2nd column* with zeros. To accomplish this: to the 2nd *row*, add 2 times the 3rd *row;* and to the 4th *row*, add (-3) times the 3rd *row.* Leave the 1st row and the 3rd row unchanged. Hence

$$\begin{vmatrix} 6 & 0 & 8 & 5 \\ -6 & -2 & -5 & 8 \\ 5 & 1 & 4 & -6 \\ 22 & 3 & 19 & -18 \end{vmatrix} = \begin{vmatrix} 6 & 0 & 8 & 5 \\ 4 & 0 & 3 & -4 \\ 5 & 1 & 4 & -6 \\ 7 & 0 & 7 & 0 \end{vmatrix}.$$

Expanding by minors according to the 2nd column, we get

$$1(-)\begin{vmatrix} 6 & 8 & 5 \\ 4 & 3 & -4 \\ 7 & 7 & 0 \end{vmatrix}.$$

In order to get zeros in the 3rd row, add to the 2nd column (-1) times the 1st column to get

$$(-1)\begin{vmatrix} 6 & 2 & 5 \\ 4 & -1 & -4 \\ 7 & 0 & 0 \end{vmatrix}.$$

Expand according to the 3rd row to get

$$(-1)7(+)\begin{vmatrix} 2 & 5 \\ -1 & -4 \end{vmatrix} = (-7)(-8+5) = 21.$$

Comments. 1. It is good news to find *0*'s and *1*'s in a determinant. Take advantage of them.

2. If a determinant does not contain an element that is 1 or -1, use property 7 to get such an element.

3. Use property 5 to simplify the computation.

Exercise 92

1. For each of the properties 1–7, make up a second-order determinant that will verify it.

* Another good selection is the 3rd row, but it contains no zero as the 2nd column does. Had we selected the 1st row, it would be necessary either to employ fractions or to obtain a 1 (1st column minus fourth column).

2. For each of the properties 1–7, make up a third-order determinant that will verify it.

For each of the following determinants (a) *expand by minors according to the 2nd column,* (b) *complete the evaluation,* (c) *check your result by expanding by minors according to the 3rd row.*

3. $\begin{vmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ -1 & 0 & 2 \end{vmatrix}.$

4. $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}.$

Evaluate the following determinants.

5. $\begin{vmatrix} 8 & 3 & 4 & -7 \\ 0 & 0 & 5 & 0 \\ 3 & 1 & 6 & -2 \\ -5 & -2 & 7 & 3 \end{vmatrix}.$

6. $\begin{vmatrix} 5 & 2 & 7 & 6 \\ 3 & 3 & 7 & 8 \\ 0 & 1 & 3 & 0 \\ 2 & 3 & 9 & 7 \end{vmatrix}.$

7. $\begin{vmatrix} 2 & 3 & -1 & 2 \\ -4 & -8 & 3 & -3 \\ 5 & 2 & 0 & 7 \\ 7 & 2 & 0 & 5 \end{vmatrix}.$

8. $\begin{vmatrix} 2 & 0 & -1 & 0 \\ -3 & 7 & 3 & 2 \\ -3 & 3 & 2 & 3 \\ 6 & 7 & -3 & 2 \end{vmatrix}.$

9. $\begin{vmatrix} 2 & -6 & 7 & 5 \\ 3 & -8 & 2 & 2 \\ -1 & 3 & 0 & 0 \\ 2 & -3 & 5 & 7 \end{vmatrix}.$

10. $\begin{vmatrix} 2 & -3 & -1 & 3 \\ -1 & 7 & 2 & -6 \\ 2 & 2 & 0 & 5 \\ 2 & 4 & 0 & 3 \end{vmatrix}.$

11. $\begin{vmatrix} 2 & 1 & 3 & -5 \\ 4 & 0 & 6 & 9 \\ -5 & -4 & -5 & 17 \\ 5 & 2 & 6 & -7 \end{vmatrix}.$

12. $\begin{vmatrix} 2 & 1 & -8 & -1 \\ 0 & -1 & 2 & 3 \\ 3 & 2 & 3 & -6 \\ 7 & -3 & 8 & 7 \end{vmatrix}.$

13. $\begin{vmatrix} 2 & 1 & -3 & 4 & -4 \\ 2 & 0 & -1 & 6 & -1 \\ 1 & 0 & 0 & -2 & -1 \\ 9 & 4 & 4 & 21 & -11 \\ 2 & 0 & 3 & -4 & 1 \end{vmatrix}.$

14. $\begin{vmatrix} 3 & 6 & 7 & 47 & -6 \\ 2 & 4 & 2 & 57 & -3 \\ -1 & 0 & 0 & 67 & 2 \\ -2 & 9 & 2 & 77 & 7 \\ 0 & 0 & 0 & 11 & 0 \end{vmatrix}.$

15. $\begin{vmatrix} 3 & 5 & 6 & -7 \\ 4 & 3 & -4 & 6 \\ 3 & 2 & 2 & -3 \\ 2 & 5 & 3 & -8 \end{vmatrix}.$

16. $\begin{vmatrix} 0 & 4 & 5 & 10 \\ 0 & 6 & 7 & 11 \\ 0 & 0 & 0 & 2 \\ 3 & 8 & 9 & 12 \end{vmatrix}.$

17. $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}.$

18. Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)$.

19. Use the definition in Art. 176 to write out the expansion of the adjoining determinant. $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

181. Solving n linear equations in n unknowns by use of determinants. We shall prove that Cramer's rule (Art. 43) enables us to solve any system of linear equations in which the number of equations is equal to the number of unknowns, provided the determinant of the coefficients is not zero.

For the sake of simplicity in notation, we give the details of the proof for $n = 4$. The reasoning, however, is general and applies to any value of n. Consider the system

$$(1) \qquad \begin{cases} a_1x + b_1y + c_1z + d_1w = k_1, \\ a_2x + b_2y + c_2z + d_2w = k_2, \\ a_3x + b_3y + c_3z + d_3w = k_3, \\ a_4x + b_4y + c_4z + d_4w = k_4. \end{cases}$$

Let D designate the determinant of the coefficients, i.e.,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Let K_x represent the determinant obtained from D by replacing the coefficients of x with the constant terms k_1, k_2, k_3, k_4. Hence

$$K_x = \begin{vmatrix} k_1 & b_1 & c_1 & d_1 \\ k_2 & b_2 & c_2 & d_2 \\ k_3 & b_3 & c_3 & d_3 \\ k_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

Assume $D \neq 0$. To solve for x, multiply the first equation of (1) by A_1,* the second by $-A_2$, the third by A_3, and the fourth by $-A_4$:

$$\begin{aligned} a_1A_1x + b_1A_1y + c_1A_1z + d_1A_1w &= k_1A_1, \\ -a_2A_2x - b_2A_2y - c_2A_2z - d_2A_2w &= -k_2A_2, \\ a_3A_3x + b_3A_3y + c_3A_3z + d_3A_3w &= k_3A_3, \\ -a_4A_4x - b_4A_4y - c_4A_4z - d_4A_4w &= -k_4A_4. \end{aligned}$$

* A_1 is the minor of a_1.

Upon adding these equations, we find that the coefficient of x is

$$a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4, \tag{2}$$

which is the expansion of D by minors according to the first column. The coefficient of y is

$$b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4,$$

which, by the corollary of Art. 179, is equal to 0. For the same reason, the coefficients of z and w are also equal to 0. Hence the result of adding the equations is

$$D \cdot x = k_1A_1 - k_2A_2 + k_3A_3 - k_4A_4. \tag{3}$$

We now see that the right side of (3) is the same as (2) except that the a's are replaced by the corresponding k's. Hence the right side of (3) is obtainable from D by replacing the a's with the corresponding k's. Therefore the right side of (3) is K_x and we have

$$D \cdot x = K_x.$$

If $D \neq 0$,

$$x = \frac{K_x}{D}. \tag{4}$$

Similarly

$$y = \frac{K_y}{D}, \quad z = \frac{K_z}{D}, \quad w = \frac{K_w}{D} \tag{5}$$

where K_y is the determinant obtained from D by replacing the coefficients of y by the constant terms, K_z is obtained by replacing the coefficients of z (i.e., column 3) by the k's, etc.

We have shown that if the system has a solution, it is given by equations (4) and (5). The proof may be completed by showing by substitution that this set of values actually satisfies each of the four equations of the system. We omit this substitution.*

For the convenience of the student, we restate **Cramer's rule:**

In a system of n *linear equations in* n *unknowns of the form*

$$\begin{cases} a_1x + b_1y + c_1z + \cdots = k_1, \\ a_2x + b_2y + c_2z + \cdots = k_2, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_nx + b_ny + c_nz + \cdots = k_n, \end{cases}$$

* See Dresden, *Solid Analytical Geometry and Determinants*, p. 37.

if the determinant D *of the coefficients of the unknowns is not zero, then the system has only one solution. In this solution, each unknown is equal to the quotient of two determinants.*

1. *The denominator is* D, *the determinant of the coefficients.*
2. *The numerator, for any unknown, is obtained from the denominator by substituting the constant terms for the coefficients of this unknown.*

Example 1. Solve:
$$\begin{cases} 3x - 4y + z - 2w = -1, & (1)\\ 2x + 5y + z - 3w = 2, & (2)\\ 4x + 3y + z - 4w = 0, & (3)\\ 7x - 2y - 5w = -8. & (4)\end{cases}$$

Solution.

$$D = \begin{vmatrix} 3 & -4 & 1 & -2\\ 2 & 5 & 1 & -3\\ 4 & 3 & 1 & -4\\ 7 & -2 & 0 & -5\end{vmatrix} = 9.$$

$$K_x = \begin{vmatrix} -1 & -4 & 1 & -2\\ 2 & 5 & 1 & -3\\ 0 & 3 & 1 & -4\\ -8 & -2 & 0 & -5\end{vmatrix} = 18; \quad x = \frac{K_x}{D} = \frac{18}{9} = 2.$$

$$K_y = \begin{vmatrix} 3 & -1 & 1 & -2\\ 2 & 2 & 1 & -3\\ 4 & 0 & 1 & -4\\ 7 & -8 & 0 & -5\end{vmatrix} = 9; \quad y = \frac{K_y}{D} = \frac{9}{9} = 1.$$

We could use determinants to find the values of z and w. It is easier, however, to *substitute* $x = 2$, $y = 1$ in (4) to obtain $w = 4$. Upon substituting $x = 2$, $y = 1$, $w = 4$ in (1), we get $z = 5$. Hence the solution of the system is

$$x = 2, \quad y = 1, \quad z = 5, \quad w = 4.$$

It should be checked by substitution in the given equations.

Exercise 93

Solve by using determinants and check by substitution.

1. $\begin{cases} x + y - z = 0,\\ 4x + 4y + 2z = 3,\\ 5x + 2y - z = 1.\end{cases}$

2. $\begin{cases} 2x - y + z = 2, \\ 3x + 4y - z = 5, \\ 4x + 5y - z = 7. \end{cases}$

3. $\begin{cases} 3x + 2y + 6z + 7w = 0, \\ 2x + y + 3z + 2w = 5, \\ 3x + 5z + 2w = 0, \\ 2x + 7z + 5w = 0. \end{cases}$

4. $\begin{cases} x - 2y + z = 8, \\ 2x + z - 4w = 4, \\ 3x + z + 2w = 0, \\ x + y - 3w = 0. \end{cases}$

5. $\begin{cases} 2x + y - w = 8, \\ x + y + z = 6, \\ x - 3z + 2w = 0, \\ 3x - y + w = 7. \end{cases}$

6. $\begin{cases} 4x - 3y - z + 2w = 3, \\ 5x + 2y + z - 3w = 0, \\ 3x + 4y + z - 4w = -2, \\ 2x - 7y + 5w = 8. \end{cases}$

7. $\begin{cases} 2x - 5y + 4z + 3w = 2, \\ 3x - y - 7z + w = 6, \\ 4x - 3y + 2z + 2w = 3, \\ 5x + 2y - 3z - w = 3. \end{cases}$

8. $\begin{cases} x + y = 1, \\ x - 3z = 5, \\ 4y + 5w = -8, \\ 6z - w = 2. \end{cases}$

9. $\begin{cases} 3v + w + x + y + 9z = 1, \\ 2v + 2w + 3x + y + 4z = 13, \\ v + 2w + x + y + 5z = 5, \\ 5v + w + 2x + 3z = 10, \\ 4v + 2w + z = 7. \end{cases}$

10. $\begin{cases} x + 2y + 3z + 2v + w = 9, \\ 3x - z - v = 8, \\ 2x - 2z - 3w = 7, \\ x - y + 3v = 1, \\ 2x + z + v + w = 6. \end{cases}$

11. Use determinants to solve for z only:

$$\begin{cases} 2x + y + z + w = 0, \\ 5x + 2y + 3z + 2w = 7, \\ 3x + 2y + z = 0, \\ 4x + 5y + 3z = 0. \end{cases}$$

12. Use determinants to solve for y **only**:

$$\begin{cases} 3x + 2y = 2w + 1, \\ 2x + w = 3y + z + 3, \\ 4x + 2z + 7 = y + w, \\ 7x + y = 3w + 1. \end{cases}$$

182. Inconsistent and dependent equations. If a system of equations has no solution, it is said to be **inconsistent.** If a system of equations has one or more solutions, it is said to be **consistent.** If a system of equations has infinitely many solutions, it is said to be **dependent.**

We state the following facts without proof.*

In a system of n *linear equations in* n *unknowns,*

I. *If* D = 0 *and if any one (or more) of the numerator determinants is not zero, then the system is inconsistent.*

Illustration 1. For the system

$$\begin{cases} x + 2y - 5z = 9, \\ 3x - y - z = 20, \\ 4x + y - 6z = 30. \end{cases}$$

$$D = \begin{vmatrix} 1 & 2 & -5 \\ 3 & -1 & -1 \\ 4 & 1 & -6 \end{vmatrix} = 0; \quad K_x = \begin{vmatrix} 9 & 2 & -5 \\ 20 & -1 & -1 \\ 30 & 1 & -6 \end{vmatrix} = -7.$$

The values of K_y and K_z are immaterial. The system is inconsistent.

II. *If* D = 0 *and if all the numerator determinants are zero, then the system may be* † *dependent.*

Illustration 2. For the system

$$\begin{cases} x + 2y - 5z = 9, & (1) \\ 3x - y - z = 20, & (2) \\ 4x + y - 6z = 29, & (3) \end{cases}$$

$$D = 0, \quad K_x = 0, \quad K_y = 0, \quad K_z = 0.$$

* See Bôcher, *Introduction to Higher Algebra*, Chap. IV.

† It need not be dependent. For example, the following system is inconsistent:

$$\begin{cases} x + 3y + 2z = 4, \\ x + 3y + 2z = 5, \\ 2x + 6y + 4z = 10. \end{cases}$$

In order to show that this particular system has infinitely many solutions, eliminate y from equations (1) and (2), to get

$$x = z + 7.$$

Eliminating x from (1) and (2), we obtain

$$y = 2z + 1.$$

It can be shown by substitution that these values of x and y will satisfy * (3). Assign to z any arbitrary value a. Then a solution of the system is

$$x = a + 7, \quad y = 2a + 1, \quad z = a.$$

Since the number of values of a is infinite, the number of solutions of the system is infinite. One of these solutions is

$$x = 12, \quad y = 11, \quad z = 5.$$

183. Homogeneous equations. If all the terms of an equation are of the same degree, the equation is said to be **homogeneous.** It follows that a linear homogeneous equation is an equation that contains only terms of the first degree; hence no constant term can be present.

A homogeneous system such as

$$\begin{cases} a_1x + b_1y + c_1z = 0, \\ a_2x + b_2y + c_2z = 0, \\ a_3x + b_3y + c_3z = 0, \end{cases}$$

obviously has the solution

$$x = 0, \quad y = 0, \quad z = 0,$$

which is called the **trivial** solution. Frequently such a solution is of little or no importance and we seek other nontrivial solutions. Without proof † we state the following fact.

A system of n *homogeneous linear equations in* n *unknowns has nontrivial solutions if and only if* D = 0.

* If this were not the case, the system would be inconsistent.

† See Dresden, *Solid Analytical Geometry and Determinants*, p. 38.

Example 1. Solve the system

$$\begin{cases} 3x + 2y - 7z = 0, & (1) \\ x + y - 5z = 0, & (2) \\ 5x + 4y - 17z = 0. & (3) \end{cases}$$

Solution. Since the system is homogeneous, we have the trivial solution

$$x = 0, \quad y = 0, \quad z = 0.$$

To see if there are other nontrivial solutions, let us evaluate D.

$$D = \begin{vmatrix} 3 & 2 & -7 \\ 1 & 1 & -5 \\ 5 & 4 & -17 \end{vmatrix} = 0.$$

Hence there are nontrivial solutions. To obtain these, eliminate y from (1) and (2) by multiplying (2) by 2 and subtracting from (1) to get $x = -3z$. Then eliminate x from (1) and (2) to obtain $y = 8z$. Next verify that (3) is satisfied by $x = -3z$, $y = 8z$. (This is true only because $D = 0$.) For each value arbitrarily assigned to z, we can obtain corresponding values for x and y. For example, if $z = 7$, then $x = -21$, and $y = 56$. The infinite number of nontrivial solutions can be written in the form

$$x : y : z = -3 : 8 : 1.$$

184. Systems of m linear equations in n unknowns.

I. *If* $m < n$. In case there are less equations than unknowns, the number of solutions is usually infinite, but it may happen that the system is inconsistent. A general rule for solving such a system is to choose m unknowns and treat the remaining $(n - m)$ unknowns as if they were constants. Try to solve for the m chosen unknowns in terms of the others. If this can be done, we can obtain as many solutions as desired by assigning arbitrary values to the remaining unknowns.

II. *If* $m > n$. In case there are more equations than unknowns, the system is usually inconsistent, but it may happen that it is consistent. In order to determine which situation prevails, solve n of the equations for the n unknowns and substitute the solution into the remaining equations. If these equations are satisfied, the system is consistent.

Exercise 94

For each of the following systems of equations, either (a) *prove the system is inconsistent, or* (b) *find all the solutions.*

1. $\begin{cases} 2x - 5y - z = 9, \\ x + y - 4z = 1, \\ 3x - 7y - 2z = 13. \end{cases}$

2. $\begin{cases} 2x + y + z = 7, \\ 3x + 4y - 2z = 1, \\ 8x + 9y - 3z = 4. \end{cases}$

3. $\begin{cases} 5x + 2y = 3, \\ 2x + y = 1, \\ 4x - y = 2. \end{cases}$

4. $\begin{cases} 2x - 3y = 2, \\ x + 2y = 15, \\ x - y = 3. \end{cases}$

5. $\begin{cases} 4x - 6y + 2z = 5, \\ 6x - 9y + 3z = 7. \end{cases}$

6. $\begin{cases} x + y - 3z = 5, \\ x - y - z = 1. \end{cases}$

7. $\begin{cases} 2x + y + z = 3, \\ 3x + y + 2z = 5, \\ x - 2y - z = -4, \\ x + 2y - z = 0. \end{cases}$

8. $\begin{cases} x + y + z = 6, \\ 2x - y + z = 3, \\ 3x - 2y - z = -4, \\ x + 2y + z = 7. \end{cases}$

9. $\begin{cases} x + 2y - z - w = 4, \\ 3x - y + z - 4w = 5, \\ 2x + y - 3w = 5. \end{cases}$

For each of the following systems of homogeneous equations, either (a) *prove that the trivial solution is the only solution, or* (b) *find the nontrivial solutions.*

10. $\begin{cases} x + y + z = 0, \\ 3x + 2y - z = 0, \\ x - y + z = 0. \end{cases}$

11. $\begin{cases} 5x + 4y - 3z - w = 0, \\ 3x - 5y + z + w = 0, \\ x + y - z = 0, \\ 4x + y - 2z = 0. \end{cases}$

12. $\begin{cases} x + 2y - 8z = 0, \\ x - y + z = 0, \\ 4x - y - 5z = 0. \end{cases}$

13. What value of k will make the following system of equations consistent?

$$\begin{cases} x + y = 5, \\ 2x + y = k, \\ 3x - y = 7. \end{cases}$$

14. For what value of k will the following system of homogeneous equations have nontrivial solutions?

$$\begin{cases} x + y + 7z = 0, \\ 2x + y + z = 0, \\ 5x + 3y + kz = 0. \end{cases}$$

chapter 22 Partial fractions

185. Introduction. If $f(x)$ and $g(x)$ are polynomials, then $\dfrac{f(x)}{g(x)}$ is called a **rational function of** x or a **rational fraction.** In Chap. 3 we learned how to combine the sum of two or more rational fractions into a single fraction. Thus,

$$\frac{x+1}{x^2+2}+\frac{4}{3x+5}=\frac{7x^2+8x+13}{(x^2+2)(3x+5)}.^{*}$$

In more advanced mathematics, particularly calculus, it is sometimes necessary to reverse the procedure, i.e., to resolve a given fraction into the algebraic sum of several simple fractions, called **partial fractions.**

A **proper** rational fraction is one whose numerator is of lower degree than its denominator. Every improper rational fraction can be expressed as the sum of a polynomial and a proper rational fraction by dividing the denominator into the numerator until the degree of the remainder is lower than that of the denominator. For example,

$$\frac{5x^3+8x^2-10}{x^2+x-2}=5x+3+\frac{7x-4}{x^2+x-2}.$$

In order to resolve a proper rational fraction into partial fractions, we employ the following theorem whose proof is beyond the scope of this book.

* It is to be remembered that this equation is an identity. It holds true for all permissible values of x.

Theorem. Any proper rational fraction which is in lowest terms can be resolved into a sum of partial fractions of the following types.

Case 1. If the linear factor $(ax + b)$ *occurs only once as a factor of the denominator, there will correspond a partial fraction*

$$\frac{A}{ax+b}$$

where A *is a constant, and* A $\neq$ 0.

Case 2. If the linear factor $(ax + b)$ *occurs* k *times as a factor of the denominator, there will correspond* k *partial fractions*

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$$

where the A's *are constants, and* $A_k \neq 0$.

Case 3. If the quadratic factor $(ax^2 + bx + c)$ *occurs only once as a factor of the denominator, there will correspond a partial fraction*

$$\frac{Ax+B}{ax^2+bx+c}$$

where A *and* B *are constants, not both* 0.

Case 4. If the quadratic factor $(ax^2 + bx + c)$ *occurs* k *times as a factor of the denominator, there will correspond* k *partial fractions*

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$$

where the A's *and* B's *are constants and* A_k *and* B_k *are not both* 0.

Illustration. The theorem states that the fraction

$$\frac{x^9+3x+11}{5(x-6)(x-7)^3(x^2+x+8)(x^2+4)^2}$$

can be resolved into partial fractions of the following types

$$\frac{A}{x-6} + \frac{B}{x-7} + \frac{C}{(x-7)^2} + \frac{D}{(x-7)^3} + \frac{Ex+F}{x^2+x+8} + \frac{Gx+H}{x^2+4} + \frac{Ix+J}{(x^2+4)^2},$$

where the capital letters are undetermined constants.

186. *Case 1.* If the denominator contains only linear factors, none repeated.

Example 1. Resolve into partial fractions:

$$\frac{13x-17}{(x-1)(x+3)(2x-1)}.$$

Solution. Using the theorem, we assume the following identity

$$\frac{13x-17}{(x-1)(x+3)(2x-1)} \equiv \frac{A}{x-1}+\frac{B}{x+3}+\frac{C}{2x-1} \qquad (1)$$

where A, B, C are constants whose values we shall now determine by two different methods.

First method. By substitution. If (1) is cleared of fractions, we get

$$13x-17 \equiv A(x+3)(2x-1)+B(x-1)(2x-1) + C(x-1)(x+3). \qquad (2)$$

Identity (1) holds true for all permissible values of x, i.e., for all values of x except $x=1$, $x=-3$, $x=\frac{1}{2}$ (which make some denominators zero). Therefore identity (2) holds true for all values of x except $x=1$, $x=-3$, $x=\frac{1}{2}$. By the corollary of Art. 125, equation (2) holds for *all values* of x, including $x=1$, $x=-3$, $x=\frac{1}{2}$.

We shall now determine the values of A, B, C, by setting x equal to convenient values in (2).

Set $x=1$: $\quad 13-17=A(4)(1); \quad -4=4A$
$$A=-1.$$

Set $x=-3$: $\quad -39-17=B(-4)(-7); \quad -56=28B$
$$B=-2.$$

Set $x=\frac{1}{2}$: $\quad \frac{13}{2}-17=C(-\frac{1}{2})(\frac{7}{2}); \quad -\frac{21}{2}=-\frac{7}{4}C$
$$C=6.$$

Hence

$$\frac{13x-17}{(x-1)(x+3)(2x-1)} \equiv -\frac{1}{x-1}-\frac{2}{x+3}+\frac{6}{2x-1}. \qquad (3)$$

The student should check the result by adding the fractions on the right side of (3).

Second method. By equating the coefficients of like powers. Upon expanding the right side of (2), we get

$$13x - 17 \equiv A(2x^2 + 5x - 3) + B(2x^2 - 3x + 1) + C(x^2 + 2x - 3).$$

Collecting like powers of x, we obtain

$$13x - 17 \equiv (2A + 2B + C)x^2 + (5A - 3B + 2C)x + (-3A + B - 3C).$$

By the corollary of Art. 125, we can equate the coefficients of like powers of x.

$$\begin{array}{ll} \text{Coeff. of } x^2: & 0 = 2A + 2B + C. \\ \text{Coeff. of } x: & 13 = 5A - 3B + 2C. \\ \text{Constant terms:} & -17 = -3A + B - 3C. \end{array}$$

The solution of this system of equations is $A = -1$, $B = -2$, $C = 6$, which is the same as that obtained by the first method.

Comment. In the first method, instead of using the convenient values $x = 1$, $x = -3$, $x = \frac{1}{2}$, we could have used *any three values* of x. The resulting system of three equations in A, B, C, would have the same solution.

187. *Case* 2. If the denominator contains linear factors, some repeated.

Example *2*. Resolve into partial fractions:

$$\frac{x^3 - 5x + 8}{(x - 1)(x - 2)^3}.$$

Solution. Using the theorem, we assume the following identity

$$\frac{x^3 - 5x + 8}{(x - 1)(x - 2)^3} \equiv \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{(x - 2)^2} + \frac{D}{(x - 2)^3} \quad (1)$$

where A, B, C, D, are undetermined constants. Clear of fractions by multiplying through by the *lowest* common denominator, $(x - 1)(x - 2)^3$:

$$x^3 - 5x + 8 \equiv A(x - 2)^3 + B(x - 1)(x - 2)^2 + C(x - 1)(x - 2) + D(x - 1). \quad (2)$$

We shall determine the values of A, B, C, D, by using a combination of the first method and the second method of Ex. 1.

In (2), substitute the convenient values $x = 1$ and $x = 2$.

Set $x = 1$: $\quad 1 - 5 + 8 = A(-1)^3; \quad 4 = -A$

$$A = -4.$$

Set $x = 2$: $\quad 8 - 10 + 8 = D(1)$

$$D = 6.$$

This exhausts the convenient values of x. We shall determine the values of B and C by equating the coefficients of x^3 and the constant terms.* From (2),

Coeff. of x^3: $\quad 1 = A + B. \qquad (3)$

Constant terms: $\quad 8 = -8A - 4B + 2C - D. \qquad (4)$

(i.e., set $x = 0$)

Since $A = -4$, it follows from (3) that $1 = -4 + B$

$$B = 5.$$

Upon substituting $A = -4$, $B = 5$, $D = 6$ into (4), we get

$$8 = 32 - 20 + 2C - 6$$
$$2 = 2C$$
$$C = 1.$$

Hence

$$\frac{x^3 - 5x + 8}{(x-1)(x-2)^3} = \frac{-4}{x-1} + \frac{5}{x-2} + \frac{1}{(x-2)^2} + \frac{6}{(x-2)^3}.$$

Comment. After finding $A = -4$ and $D = 6$ by setting x equal to convenient values, we could obtain two more equations by setting x equal to *any other two values*. For example, if $x = 3$, then (2) becomes

$$20 = A + 2B + 2C + 2D$$

and for $x = -1$, equation (2) becomes

$$12 = -27A - 18B + 6C - 2D.$$

If these two equations are solved with $A = -4$, $D = 6$, we obtain $B = 5$, $C = 1$.

* When only two additional equations are needed, the most convenient procedure is to equate the coefficients of the highest power of x and then equate the constant terms. This can be done without actually expanding the right side of (2). We could, however, have equated the coefficients of x^2 and x:

$$0 = -6A - 5B + C,$$
$$-5 = 12A + 8B - 3C + D.$$

Exercise 95

Resolve into partial fractions.

1. $\dfrac{9x-1}{(x-4)(x+1)}$.

2. $\dfrac{7x+37}{(x-5)(2x-1)}$.

3. $\dfrac{x+2}{(3x-4)(4x+3)}$.

4. $\dfrac{x+1}{(x-1)(3x-1)x}$.

5. $\dfrac{14x^2-42x-6}{(3x-2)(x-4)(x+1)}$.

6. $\dfrac{7x^2+17}{(2x-1)(x^2-2x-3)}$.

7. $\dfrac{5x^3+8x^2-10}{x^2+x-2}$.

8. $\dfrac{x^2-7x+5}{(x-2)(x-3)(x+4)}$.

9. $\dfrac{3x^2-13x+11}{(x+2)(x-5)^2}$.

10. $\dfrac{x^2+x+1}{(x-1)(x-2)^2}$.

11. $\dfrac{5x^2-2}{(4x-1)(x-1)^2}$.

12. $\dfrac{5x+9}{(x+4)^3}$.

13. $\dfrac{7x^2-28x+19}{(x-2)^3}$.

14. $\dfrac{x^3+3x^2-4x-5}{(x^2-2x+1)(x^2-3x+2)}$.

15. $\dfrac{17}{(x-7)(x^2-14x+49)}$.

16. $\dfrac{2x^2-17x+39}{(x-5)^2}$.

17. $\dfrac{5x^3-9x^2-4x}{(x-1)^4}$.

18. $\dfrac{10x^2+32}{(x-4)^2x^3}$.

19. $\dfrac{123x-87}{(x+5)(x-1)(x-4)(2x-3)}$.

20. $\dfrac{5x^3+6x^2+7x+8}{x^4}$.

188. ***Case 3.* If the denominator contains quadratic factors, none repeated.**

Example 3. Resolve into partial fractions:

$$\frac{2x^2+13}{(x-3)(x^2+5x+7)}.$$

Solution. Using the theorem, we assume the following identity

$$\frac{2x^2+13}{(x-3)(x^2+5x+7)} \equiv \frac{A}{x-3} + \frac{Bx+C}{x^2+5x+7} \tag{1}$$

where A, B, C, are undetermined constants. Clearing of fractions, we get

$$2x^2+13 \equiv A(x^2+5x+7) + (Bx+C)(x-3). \tag{2}$$

Set $x = 3$: $\qquad 18 + 13 = A(9 + 15 + 7); \quad 31 = 31A$

$$A = 1.$$

Equate coeff. of x^2: $\qquad 2 = A + B.$

$$B = 1.$$

Equate constant terms: $13 = 7A - 3C; \quad 13 = 7 - 3C; \quad 3C = -6$
(i.e., set $x = 0$) $\qquad C = -2.$

Hence

$$\frac{2x^2 + 13}{(x - 3)(x^2 + 5x + 7)} \equiv \frac{1}{x - 3} + \frac{x - 2}{x^2 + 5x + 7}.$$

189. ***Case 4.*** **If the denominator contains quadratic factors, some repeated.**

Example 4. Resolve into partial fractions:

$$\frac{x^4 + 18x^2 - 2x + 29}{(x + 1)(x^2 + 4)^2}.$$

Solution. Using the theorem, we assume the following identity

$$\frac{x^4 + 18x^2 - 2x + 29}{(x + 1)(x^2 + 4)^2} \equiv \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{(x^2 + 4)^2} \qquad (1)$$

where A, B, C, D, E, are undetermined constants. Clear of fractions by multiplying through by the *lowest* common denominator:

$$x^4 + 18x^2 - 2x + 29 \equiv A(x^2 + 4)^2 + (Bx + C)(x + 1)(x^2 + 4) + (Dx + E)(x + 1). \qquad (2)$$

In this identity,

set $x = -1$: $\quad 1 + 18 + 2 + 29 = A(5)^2; \quad 50 = 25A$

$$A = 2.$$

The values of the remaining constants will be obtained by equating coefficients of like powers. Expand (2) and collect terms:

$$x^4 + 18x^2 - 2x + 29 \equiv (A + B)x^4 + (B + C)x^3 + (8A + 4B + C + D)x^2 + (4B + 4C + D + E)x + (16A + 4C + E).$$

Equate coefficients of like powers:

Coeff. of x^4:	$1 = A + B$
Coeff. of x^3:	$0 = B + C$
Coeff. of x^2:	$18 = 8A + 4B + C + D$
Coeff. of x:	$-2 = 4B + 4C + D + E$
Constant terms:	$29 = 16A + 4C + E.$

In *any four* of these equations, substitute $A = 2$, to get

$$B = -1, \quad C = 1, \quad D = 5, \quad E = -7.$$

Hence

$$\frac{x^4 + 18x^2 - 2x + 29}{(x+1)(x^2+4)^2} \equiv \frac{2}{x+1} + \frac{-x+1}{x^2+4} + \frac{5x-7}{(x^2+4)^2}.$$

Exercise 96

Resolve into partial fractions.

1. $\dfrac{9x^2 + x + 6}{(x-1)(x^2+3)}$.

2. $\dfrac{7x}{(x-2)(x^2+x+1)}$.

3. $\dfrac{5x^2 - 3}{(x+5)(2x^2 - x + 6)}$.

4. $\dfrac{8x^2 + 4x + 2}{(2x-1)(4x^2+5)}$.

5. $\dfrac{x^3 - 6x^2 + 2x - 4}{(x^2 - 3x + 2)(3x^2 + 4)}$.

6. $\dfrac{x^3 + 29x - 49}{(x^2 - 6x + 9)(x^2 + 4)}$.

7. $\dfrac{6x}{x^3 - 8}$.

8. $\dfrac{11x + 23}{(x-3)(x^3+1)}$.

9. $\dfrac{7x^4 - 4x^2 - x - 1}{x^3(5x^2+1)}$.

10. $\dfrac{7x^3 + 10x + 1}{(x^2+1)(x^2+2)}$.

11. $\dfrac{11x + 15}{(x^2 + 3x + 5)(x^2 + 7)}$.

12. $\dfrac{x^3 + 3}{(x^2 + x + 3)(x^2 - x + 3)}$.

13. $\dfrac{5x^3 + 6x^2 + 22x + 28}{(x^2+3)^2}$.

14. $\dfrac{4x^3 + 5}{(4x^2+7)^2}$.

15. $\dfrac{x^2}{(x^2 + x + 4)^2}$.

16. $\dfrac{3x^4 + 24x^2 + 22}{(x-1)(x^2+6)^2}$.

17. $\dfrac{3x^4 + x^2 + 6x}{(x-2)(x^2+x+2)^2}$.

18. $\dfrac{x^5 + 9x^2 - 1}{x^2(x^2 + 3x + 1)^2}$.

19. $\dfrac{3x^4 + 6x^2 + 4x + 8}{(x^2+1)^3}$.

20. $\dfrac{x^6 + 2x^3 - 9x^2 - 17}{(x^2+1)^2(x^2+2)^2}$.

Tables

No.	Square	Cube	Square root	Cube root	No.	Square	Cube	Square root	Cube root
1	1	1	1.000	1.000	51	2 601	132 651	7.141	3.708
2	4	8	1.414	1.260	52	2 704	140 608	7.211	3.733
3	9	27	1.732	1.442	53	2 809	148 877	7.280	3.756
4	16	64	2.000	1.587	54	2 916	157 464	7.348	3.780
5	25	125	2.236	1.710	55	3 025	166 375	7.416	3.803
6	36	216	2.449	1.817	56	3 136	175 616	7.483	3.826
7	49	343	2.646	1.913	57	3 249	185 193	7.550	3.849
8	64	512	2.828	2.000	58	3 364	195 112	7.616	3.871
9	81	729	3.000	2.080	59	3 481	205 379	7.681	3.893
10	100	1 000	3.162	2.154	**60**	3 600	216 000	7.746	3.915
11	121	1 331	3.317	2.224	61	3 721	226 981	7.810	3.936
12	144	1 728	3.464	2.289	62	3 844	238 328	7.874	3.958
13	169	2 197	3.606	2.351	63	3 969	250 047	7.937	3.979
14	196	2 744	3.742	2.410	64	4 096	262 144	8.000	4.000
15	225	3 375	3.873	2.466	65	4 225	274 625	8.062	4.021
16	256	4 096	4.000	2.520	66	4 356	287 496	8.124	4.041
17	289	4 913	4.123	2.571	67	4 489	300 763	8.185	4.062
18	324	5 832	4.243	2.621	68	4 624	314 432	8.246	4.082
19	361	6 859	4.359	2.668	69	4 761	328 509	8.307	4.102
20	400	8 000	4.472	2.714	**70**	4 900	343 000	8.367	4.121
21	441	9 261	4.583	2.759	71	5 041	357 911	8.426	4.141
22	484	10 648	4.690	2.802	72	5 184	373 248	8.485	4.160
23	529	12 167	4.796	2.844	73	5 329	389 017	8.544	4.179
24	576	13 824	4.899	2.884	74	5 476	405 224	8.602	4.198
25	625	15 625	5.000	2.924	75	5 625	421 875	8.660	4.217
26	676	17 576	5.099	2.962	76	5 776	438 976	8.718	4.236
27	729	19 683	5.196	3.000	77	5 929	456 533	8.775	4.254
28	784	21 952	5.292	3.037	78	6 084	474 552	8.832	4.273
29	841	24 389	5.385	3.072	79	6 241	493 039	8.888	4.291
30	900	27 000	5.477	3.107	**80**	6 400	512 000	8.944	4.309
31	961	29 791	5.568	3.141	81	6 561	531 441	9.000	4.327
32	1 024	32 768	5.657	3.175	82	6 724	551 368	9.055	4.344
33	1 089	35 937	5.745	3.208	83	6 889	571 787	9.110	4.362
34	1 156	39 304	5.831	3.240	84	7 056	592 704	9.165	4.380
35	1 225	42 875	5.916	3.271	85	7 225	614 125	9.220	4.397
36	1 296	46 656	6.000	3.302	86	7 396	636 056	9.274	4.414
37	1 369	50 653	6.083	3.332	87	7 569	658 503	9.327	4.431
38	1 444	54 872	6.164	3.362	88	7 744	681 472	9.381	4.448
39	1 521	59 319	6.245	3.391	89	7 921	704 969	9.434	4.465
40	1 600	64 000	6.325	3.420	**90**	8 100	729 000	9.487	4.481
41	1 681	68 921	6.403	3.448	91	8 281	753 571	9.539	4.498
42	1 764	74 088	6.481	3.476	92	8 464	778 688	9.592	4.514
43	1 849	79 507	6.557	3.503	93	8 649	804 357	9.644	4.531
44	1 936	85 184	6.633	3.530	94	8 836	830 584	9.695	4.547
45	2 025	91 125	6.708	3.557	95	9 025	857 375	9.747	4.563
46	2 116	97 336	6.782	3.583	96	9 216	884 736	9.798	4.579
47	2 209	103 823	6.856	3.609	97	9 409	912 673	9.849	4.595
48	2 304	110 592	6.928	3.634	98	9 604	941 192	9.899	4.610
49	2 401	117 649	7.000	3.659	99	9 801	970 299	9.950	4.626
50	2 500	125 000	7.071	3.684	**100**	10 000	1 000 000	10.000	4.642

TABLE II.—TRIGONOMETRIC FUNCTIONS

Angle	Sin	Cos	Tan	Angle	Sin	Cos	Tan
0°	.000	1.000	.000	45°	.707	.707	1.000
1°	.018	.999	.018	46°	.719	.695	1.036
2°	.035	.999	.035	47°	.731	.682	1.072
3°	.052	.998	.052	48°	.743	.669	1.111
4°	.070	.997	.070	49°	.755	.656	1.150
5°	.087	.996	.087	50°	.766	.643	1.192
6°	.105	.994	.105	51°	.777	.629	1.235
7°	.122	.992	.123	52°	.788	.616	1.280
8°	.139	.990	.141	53°	.799	.602	1.327
9°	.156	.988	.158	54°	.809	.588	1.376
10°	.174	.985	.176	55°	.819	.574	1.428
11°	.191	.982	.194	56°	.829	.559	1.483
12°	.208	.978	.213	57°	.839	.545	1.540
13°	.225	.974	.231	58°	.848	.530	1.600
14°	.242	.970	.249	59°	.857	.515	1.664
15°	.259	.966	.268	60°	.866	.500	1.732
16°	.276	.961	.287	61°	.875	.485	1.804
17°	.292	.956	.306	62°	.883	.469	1.881
18°	.309	.951	.325	63°	.891	.454	1.963
19°	.326	.946	.344	64°	.899	.438	2.050
20°	.342	.940	.364	65°	.906	.423	2.144
21°	.358	.934	.384	66°	.914	.407	2.246
22°	.375	.927	.404	67°	.921	.391	2.356
23°	.391	.921	.424	68°	.927	.375	2.475
24°	.407	.914	.445	69°	.934	.358	2.605
25°	.423	.906	.466	70°	.940	.342	2.747
26°	.438	.899	.488	71°	.946	.326	2.904
27°	.454	.891	.510	72°	.951	.309	3.078
28°	.469	.883	.532	73°	.956	.292	3.271
29°	.485	.875	.554	74°	.961	.276	3.487
30°	.500	.866	.577	75°	.966	.259	3.732
31°	.515	.857	.601	76°	.970	.242	4.011
32°	.530	.848	.625	77°	.974	.225	4.331
33°	.545	.839	.649	78°	.978	.208	4.705
34°	.559	.829	.675	79°	.982	.191	5.145
35°	.574	.819	.700	80°	.985	.174	5.671
36°	.588	.809	.727	81°	.988	.156	6.314
37°	.602	.799	.754	82°	.990	.139	7.115
38°	.616	.788	.781	83°	.992	.122	8.144
39°	.629	.777	.810	84°	.994	.105	9.514
40°	.643	.766	.839	85°	.996	.087	11.430
41°	.656	.755	.869	86°	.997	.070	14.300
42°	.669	.743	.900	87°	.998	.052	19.081
43°	.682	.731	.933	88°	.999	.035	28.636
44°	.695	.719	.966	89°	.999	.018	57.290
				90°	1.000	.000	——

TABLE III.—MANTISSAS OF COMMON LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

TABLE III.—MANTISSAS OF COMMON LOGARITHMS (*Continued*)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

TABLE IV.—COMPOUND AMOUNT: $(1 + i)^n$

n	1%	1½%	2%	2½%	3%	4%	5%	6%	7%
1	1.0100	1.0150	1.0200	1.0250	1.0300	1.0400	1.0500	1.0600	1.0700
2	1.0201	1.0302	1.0404	1.0506	1.0609	1.0816	1.1025	1.1236	1.1449
3	1.0303	1.0457	1.0612	1.0769	1.0927	1.1249	1.1576	1.1910	1.2250
4	1.0406	1.0614	1.0824	1.1038	1.1255	1.1699	1.2155	1.2625	1.3108
5	1.0510	1.0773	1.1041	1.1314	1.1593	1.2167	1.2763	1.3382	1.4026
6	1.0615	1.0934	1.1262	1.1597	1.1941	1.2653	1.3401	1.4185	1.5007
7	1.0721	1.1098	1.1487	1.1887	1.2299	1.3159	1.4071	1.5036	1.6058
8	1.0829	1.1265	1.1717	1.2184	1.2668	1.3686	1.4775	1.5938	1.7182
9	1.0937	1.1434	1.1951	1.2489	1.3048	1.4233	1.5513	1.6895	1.8385
10	1.1046	1.1605	1.2190	1.2801	1.3439	1.4802	1.6289	1.7908	1.9672
11	1.1157	1.1779	1.2434	1.3121	1.3842	1.5395	1.7103	1.8983	2.1049
12	1.1268	1.1956	1.2682	1.3449	1.4258	1.6010	1.7959	2.0122	2.2522
13	1.1381	1.2136	1.2936	1.3785	1.4685	1.6651	1.8856	2.1329	2.4098
14	1.1495	1.2318	1.3195	1.4130	1.5126	1.7317	1.9799	2.2609	2.5785
15	1.1610	1.2502	1.3459	1.4483	1.5580	1.8009	2.0789	2.3966	2.7590
16	1.1726	1.2690	1.3728	1.4845	1.6047	1.8730	2.1829	2.5404	2.9522
17	1.1843	1.2880	1.4002	1.5216	1.6528	1.9479	2.2920	2.6928	3.1588
18	1.1961	1.3073	1.4282	1.5597	1.7024	2.0258	2.4066	2.8543	3.3799
19	1.2081	1.3270	1.4568	1.5987	1.7535	2.1068	2.5270	3.0256	3.6165
20	1.2202	1.3469	1.4859	1.6386	1.8061	2.1911	2.6533	3.2071	3.8697
21	1.2324	1.3671	1.5157	1.6796	1.8603	2.2788	2.7860	3.3996	4.1406
22	1.2447	1.3876	1.5460	1.7216	1.9161	2.3699	2.9253	3.6035	4.4304
23	1.2572	1.4084	1.5769	1.7646	1.9736	2.4647	3.0715	3.8197	4.7405
24	1.2697	1.4295	1.6084	1.8087	2.0328	2.5633	3.2251	4.0489	5.0724
25	1.2824	1.4509	1.6406	1.8539	2.0938	2.6658	3.3864	4.2919	5.4274
26	1.2953	1.4727	1.6734	1.9003	2.1566	2.7725	3.5557	4.5494	5.8074
27	1.3082	1.4948	1.7069	1.9478	2.2213	2.8834	3.7335	4.8223	6.2139
28	1.3213	1.5172	1.7410	1.9965	2.2879	2.9987	3.9201	5.1117	6.6488
29	1.3345	1.5400	1.7758	2.0464	2.3566	3.1187	4.1161	5.4184	7.1143
30	1.3478	1.5631	1.8114	2.0976	2.4273	3.2434	4.3219	5.7435	7.6123
31	1.3613	1.5865	1.8476	2.1500	2.5001	3.3731	4.5380	6.0881	8.1451
32	1.3749	1.6103	1.8845	2.2038	2.5751	3.5081	4.7649	6.4534	8.7153
33	1.3887	1.6345	1.9222	2.2589	2.6523	3.6484	5.0032	6.8406	9.3253
34	1.4026	1.6590	1.9607	2.3153	2.7319	3.7943	5.2533	7.2510	9.9781
35	1.4166	1.6839	1.9999	2.3732	2.8139	3.9461	5.5160	7.6861	10.6766
36	1.4308	1.7091	2.0399	2.4325	2.8983	4.1039	5.7918	8.1473	11.4239
37	1.4451	1.7348	2.0807	2.4933	2.9852	4.2681	6.0814	8.6361	12.2236
38	1.4595	1.7608	2.1223	2.5557	3.0748	4.4388	6.3855	9.1543	13.0793
39	1.4741	1.7872	2.1647	2.6196	3.1670	4.6164	6.7048	9.7035	13.9948
40	1.4889	1.8140	2.2080	2.6851	3.2620	4.8010	7.0400	10.2857	14.9745
41	1.5038	1.8412	2.2522	2.7522	3.3599	4.9931	7.3920	10.9029	16.0227
42	1.5188	1.8688	2.2972	2.8210	3.4607	5.1928	7.7616	11.5570	17.1443
43	1.5340	1.8969	2.3432	2.8915	3.5645	5.4005	8.1497	12.2505	18.3444
44	1.5493	1.9253	2.3901	2.9638	3.6715	5.6165	8.5572	12.9855	19.6285
45	1.5648	1.9542	2.4379	3.0379	3.7816	5.8412	8.9850	13.7646	21.0025
46	1.5805	1.9835	2.4866	3.1139	3.8950	6.0748	9.4343	14.5905	22.4726
47	1.5963	2.0133	2.5363	3.1917	4.0119	6.3178	9.9060	15.4659	24.0457
48	1.6122	2.0435	2.5871	3.2715	4.1323	6.5705	10.4013	16.3939	25.7289
49	1.6283	2.0741	2.6388	3.3533	4.2562	6.8333	10.9213	17.3775	27.5299
50	1.6446	2.1052	2.6916	3.4371	4.3839	7.1067	11.4674	18.4202	29.4570

TABLE V.—PRESENT VALUE: $(1+i)^{-n}$

n	1%	$1\frac{1}{2}$%	2%	$2\frac{1}{2}$%	3%	4%	5%	6%	7%
1	.99010	.98522	.98039	.97561	.97087	.96154	.95238	.94340	.93458
2	.98030	.97066	.96117	.95181	.94260	.92456	.90703	.89000	.87344
3	.97059	.95632	.94232	.92860	.91514	.88900	.86384	.83962	.81630
4	.96098	.94218	.92385	.90595	.88849	.85480	.82270	.79209	.76290
5	.95147	.92826	.90573	.88385	.86261	.82193	.78353	.74726	.71299
6	.94205	.91454	.88797	.86230	.83748	.79031	.74622	.70496	.66634
7	.93272	.90103	.87056	.84127	.81309	.75992	.71068	.66506	.62275
8	.92348	.88771	.85349	.82075	.78941	.73069	.67684	.62741	.58201
9	.91434	.87459	.83676	.80073	.76642	.70259	.64461	.59190	.54393
10	.90529	.86167	.82035	.78120	.74409	.67556	.61391	.55839	.50835
11	.89632	.84893	.80426	.76214	.72242	.64958	.58468	.52679	.47509
12	.88745	.83639	.78849	.74356	.70138	.62460	.55684	.49697	.44401
13	.87866	.82403	.77303	.72542	.68095	.60057	.53032	.46884	.41496
14	.86996	.81185	.75788	.70773	.66112	.57748	.50507	.44230	.38782
15	.86135	.79985	.74301	.69047	.64186	.55526	.48102	.41727	.36245
16	.85282	.78803	.72845	.67362	.62317	.53391	.45811	.39365	.33873
17	.84438	.77639	.71416	.65720	.60502	.51337	.43630	.37136	.31657
18	.83602	.76491	.70016	.64117	.58739	.49363	.41552	.35034	.29586
19	.82774	.75361	.68643	.62553	.57029	.47464	.39573	.33051	.27651
20	.81954	.74247	.67297	.61027	.55368	.45639	.37689	.31180	.25842
21	.81143	.73150	.65978	.59539	.53755	.43883	.35894	.29416	.24151
22	.80340	.72069	.64684	.58086	.52189	.42196	.34185	.27751	.22571
23	.79544	.71004	.63416	.56670	.50669	.40573	.32557	.26180	.21095
24	.78757	.69954	.62172	.55288	.49193	.39012	.31007	.24698	.19715
25	.77977	.68921	.60953	.53939	.47761	.37512	.29530	.23300	.18425
26	.77205	.67902	.59758	.52623	.46369	.36069	.28124	.21981	.17220
27	.76440	.66899	.58586	.51340	.45019	.34682	.26785	.20737	.16093
28	.75684	.65910	.57437	.50088	.43708	.33348	.25509	.19563	.15040
29	.74934	.64936	.56311	.48866	.42435	.32065	.24295	.18456	.14056
30	.74192	.63976	.55207	.47674	.41199	.30832	.23138	.17411	.13137
31	.73458	.63031	.54125	.46511	.39999	.29646	.22036	.16425	.12277
32	.72730	.62099	.53063	.45377	.38834	.28506	.20987	.15496	.11474
33	.72010	.61182	.52023	.44270	.37703	.27409	.19987	.14619	.10723
34	.71297	.60277	.51003	.43191	.36604	.26355	.19035	.13791	.10022
35	.70591	.59387	.50003	.42137	.35538	.25342	.18129	.13011	.09366
36	.69892	.58509	.49022	.41109	.34503	.24367	.17266	.12274	.08754
37	.69200	.57644	.48061	.40107	.33498	.23430	.16444	.11579	.08181
38	.68515	.56792	.47119	.39128	.32523	.22529	.15661	.10924	.07646
39	.67837	.55953	.46195	.38174	.31575	.21662	.14915	.10306	.07146
40	.67165	.55126	.45289	.37243	.30656	.20829	.14205	.09722	.06678
41	.66500	.54312	.44401	.36335	.29763	.20028	.13528	.09172	.06241
42	.65842	.53509	.43530	.35448	.28896	.19257	.12884	.08653	.05833
43	.65190	.52718	.42677	.34584	.28054	.18517	.12270	.08163	.05451
44	.64545	.51939	.41840	.33740	.27237	.17805	.11686	.07701	.05095
45	.63905	.51171	.41020	.32917	.26444	.17120	.11130	.07265	.04761
46	.63273	.50415	.40215	.32115	.25674	.16461	.10600	.06854	.04450
47	.62646	.49670	.39427	.31331	.24926	.15828	.10095	.06466	.04159
48	.62026	.48936	.38654	.30567	.24200	.15219	.09614	.06100	.03887
49	.61412	.48213	.37896	.29822	.23495	.14634	.09156	.05755	.03632
50	.60804	.47500	.37153	.29094	.22811	.14071	.08720	.05429	.03395

TABLE VI.—AMOUNT OF AN ANNUITY: $s_{\overline{n}|} = \dfrac{(1+i)^n - 1}{i}$

n	1%	$1\frac{1}{2}$%	2%	$2\frac{1}{2}$%	3%	4%	5%	6%	7%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0150	2.0200	2.0250	2.0300	2.0400	2.0500	2.0600	2.0700
3	3.0301	3.0452	3.0604	3.0756	3.0909	3.1216	3.1525	3.1836	3.2149
4	4.0604	4.0909	4.1216	4.1525	4.1836	4.2465	4.3101	4.3746	4.4399
5	5.1010	5.1523	5.2040	5.2563	5.3091	5.4163	5.5256	5.6371	5.7507
6	6.1520	6.2296	6.3081	6.3877	6.4684	6.6330	6.8019	6.9753	7.1533
7	7.2135	7.3230	7.4343	7.5474	7.6625	7.8983	8.1420	8.3938	8.6540
8	8.2857	8.4328	8.5830	8.7361	8.8923	9.2142	9.5491	9.8975	10.2598
9	9.3685	9.5593	9.7546	9.9545	10.1591	10.5828	11.0266	11.4913	11.9780
10	10.4622	10.7027	10.9497	11.2034	11.4639	12.0061	12.5779	13.1808	13.8164
11	11.5668	11.8633	12.1687	12.4835	12.8078	13.4864	14.2068	14.9716	15.7836
12	12.6825	13.0412	13.4121	13.7956	14.1920	15.0258	15.9171	16.8699	17.8885
13	13.8093	14.2368	14.6803	15.1404	15.6178	16.6268	17.7130	18.8821	20.1406
14	14.9474	15.4504	15.9739	16.5190	17.0863	18.2919	19.5986	21.0151	22.5505
15	16.0969	16.6821	17.2934	17.9319	18.5989	20.0236	21.5786	23.2760	25.1290
16	17.2579	17.9324	18.6393	19.3802	20.1569	21.8245	23.6575	25.6725	27.8881
17	18.4304	19.2014	20.0121	20.8647	21.7616	23.6975	25.8404	28.2129	30.8402
18	19.6147	20.4894	21.4123	22.3863	23.4144	25.6454	28.1324	30.9057	33.9990
19	20.8109	21.7967	22.8406	23.9460	25.1169	27.6712	30.5390	33.7600	37.3790
20	22.0190	23.1237	24.2974	25.5447	26.8704	29.7781	33.0660	36.7856	40.9955
21	23.2392	24.4705	25.7833	27.1833	28.6765	31.9692	35.7193	39.9927	44.8652
22	24.4716	25.8376	27.2990	28.8629	30.5368	34.2480	38.5052	43.3923	49.0057
23	25.7163	27.2251	28.8450	30.5844	32.4529	36.6179	41.4305	46.9958	53.4361
24	26.9735	28.6335	30.4219	32.3490	34.4265	39.0826	44.5020	50.8156	58.1767
25	28.2432	30.0630	32.0303	34.1578	36.4593	41.6459	47.7271	54.8645	63.2490
26	29.5256	31.5140	33.6709	36.0117	38.5530	44.3117	51.1135	59.1564	68.6765
27	30.8209	32.9867	35.3443	37.9120	40.7096	47.0842	54.6691	63.7058	74.4838
28	32.1291	34.4815	37.0512	39.8598	42.9309	49.9676	58.4026	68.5281	80.6977
29	33.4504	35.9987	38.7922	41.8563	45.2189	52.9663	62.3227	73.6398	87.3465
30	34.7849	37.5387	40.5681	43.9027	47.5754	56.0849	66.4388	79.0582	94.4608
31	36.1327	39.1018	42.3794	46.0003	50.0027	59.3283	70.7608	84.8017	102.0730
32	37.4941	40.6883	44.2270	48.1503	52.5028	62.7015	75.2988	90.8898	110.2182
33	38.8690	42.2986	46.1116	50.3540	55.0778	66.2095	80.0638	97.3432	118.9334
34	40.2577	43.9331	48.0338	52.6129	57.7302	69.8579	85.0670	104.1838	128.2588
35	41.6603	45.5921	49.9945	54.9282	60.4621	73.6522	90.3203	111.4348	138.2369
36	43.0769	47.2760	51.9944	57.3014	63.2759	77.5983	95.8363	119.1209	148.9135
37	44.5076	48.9851	54.0343	59.7339	66.1742	81.7022	101.6281	127.2681	160.3374
38	45.9527	50.7199	56.1149	62.2273	69.1594	85.9703	107.7095	135.9042	172.5610
39	47.4123	52.4807	58.2372	64.7830	72.2342	90.4091	114.0950	145.0585	185.6403
40	48.8864	54.2679	60.4020	67.4026	75.4013	95.0255	120.7998	154.7620	199.6351
41	50.3752	56.0819	62.6100	70.0876	78.6633	99.8265	127.8398	165.0477	214.6096
42	51.8790	57.9231	64.8622	72.8398	82.0232	104.8196	135.2318	175.9505	230.6322
43	53.3978	59.7920	67.1595	75.6608	85.4839	110.0124	142.9933	187.5076	247.7765
44	54.9318	61.6889	69.5027	78.5523	89.0484	115.4129	151.1430	199.7580	266.1209
45	56.4811	63.6142	71.8927	81.5161	92.7199	121.0294	159.7002	212.7435	285.7493
46	58.0459	65.5684	74.3306	84.5540	96.5015	126.8706	168.6852	226.5081	306.7518
47	59.6263	67.5519	76.8172	87.6679	100.3965	132.9454	178.1194	241.0986	329.2244
48	61.2226	69.5652	79.3535	90.8596	104.4084	139.2632	188.0254	256.5645	353.2701
49	62.8348	71.6087	81.9406	94.1311	108.5406	145.8337	198.4267	272.9584	378.9990
50	64.4632	73.6828	84.5794	97.4843	112.7969	152.6671	209.3480	290.3359	406.5289

TABLE VII.—PRESENT VALUE OF AN ANNUITY: $a_{\overline{n}|} = \dfrac{1 - (1 + i)^{-n}}{i}$

n	1%	$1\frac{1}{2}$%	2%	$2\frac{1}{2}$%	3%	4%	5%	6%	7%
1	.9901	.9852	.9804	.9756	.9709	.9615	.9524	.9434	.9346
2	1.9704	1.9559	1.9416	1.9274	1.9135	1.8861	1.8594	1.8334	1.8080
3	2.9410	2.9122	2.8839	2.8560	2.8286	2.7751	2.7232	2.6730	2.6243
4	3.9020	3.8544	3.8077	3.7620	3.7171	3.6299	3.5460	3.4651	3.3872
5	4.8534	4.7826	4.7135	4.6458	4.5797	4.4518	4.3295	4.2124	4.1002
6	5.7955	5.6972	5.6014	5.5081	5.4172	5.2421	5.0757	4.9173	4.7665
7	6.7282	6.5982	6.4720	6.3494	6.2303	6.0021	5.7864	5.5824	5.3893
8	7.6517	7.4859	7.3255	7.1701	7.0197	6.7327	6.4632	6.2098	5.9713
9	8.5660	8.3605	8.1622	7.9709	7.7861	7.4353	7.1078	6.8017	6.5152
10	9.4713	9.2222	8.9826	8.7521	8.5302	8.1109	7.7217	7.3601	7.0236
11	10.3676	10.0711	9.7868	9.5142	9.2526	8.7605	8.3064	7.8869	7.4987
12	11.2551	10.9075	10.5753	10.2578	9.9540	9.3851	8.8633	8.3838	7.9427
13	12.1337	11.7315	11.3484	10.9832	10.6350	9.9856	9.3936	8.8527	8.3577
14	13.0037	12.5434	12.1062	11.6909	11.2961	10.5631	9.8986	9.2950	8.7455
15	13.8651	13.3432	12.8493	12.3814	11.9379	11.1184	10.3797	9.7122	9.1079
16	14.7179	14.1313	13.5777	13.0550	12.5611	11.6523	10.8378	10.1059	9.4466
17	15.5623	14.9076	14.2919	13.7122	13.1661	12.1657	11.2741	10.4773	9.7632
18	16.3983	15.6726	14.9920	14.3534	13.7535	12.6593	11.6896	10.8276	10.0591
19	17.2260	16.4262	15.6785	14.9789	14.3238	13.1339	12.0853	11.1581	10.3356
20	18.0456	17.1686	16.3514	15.5892	14.8775	13.5903	12.4622	11.4699	10.5940
21	18.8570	17.9001	17.0112	16.1845	15.4150	14.0292	12.8212	11.7641	10.8355
22	19.6604	18.6208	17.6580	16.7654	15.9369	14.4511	13.1630	12.0416	11.0612
23	20.4558	19.3309	18.2922	17.3321	16.4436	14.8568	13.4886	12.3034	11.2722
24	21.2434	20.0304	18.9139	17.8850	16.9355	15.2470	13.7986	12.5504	11.4693
25	22.0232	20.7196	19.5235	18.4244	17.4131	15.6221	14.0939	12.7834	11.6536
26	22.7952	21.3986	20.1210	18.9506	17.8768	15.9828	14.3752	13.0032	11.8258
27	23.5596	22.0676	20.7069	19.4640	18.3270	16.3296	14.6430	13.2105	11.9867
28	24.3164	22.7267	21.2813	19.9649	18.7641	16.6631	14.8981	13.4062	12.1371
29	25.0658	23.3761	21.8444	20.4535	19.1885	16.9837	15.1411	13.5907	12.2777
30	25.8077	24.0158	22.3965	20.9303	19.6004	17.2920	15.3725	13.7648	12.4090
31	26.5423	24.6461	22.9377	21.3954	20.0004	17.5885	15.5928	13.9291	12.5318
32	27.2696	25.2671	23.4683	21.8492	20.3888	17.8736	15.8027	14.0840	12.6466
33	27.9897	25.8790	23.9886	22.2919	20.7658	18.1476	16.0025	14.2302	12.7538
34	28.7027	26.4817	24.4986	22.7238	21.1318	18.4112	16.1929	14.3681	12.8540
35	29.4086	27.0756	24.9986	23.1452	21.4872	18.6646	16.3742	14.4982	12.9477
36	30.1075	27.6607	25.4888	23.5563	21.8323	18.9083	16.5469	14.6210	13.0352
37	30.7995	28.2371	25.9695	23.9573	22.1672	19.1426	16.7113	14.7368	13.1170
38	31.4847	28.8051	26.4406	24.3486	22.4925	19.3679	16.8679	14.8460	13.1935
39	32.1630	29.3646	26.9026	24.7303	22.8082	19.5845	17.0170	14.9491	13.2649
40	32.8347	29.9158	27.3555	25.1028	23.1148	19.7928	17.1591	15.0463	13.3317
41	33.4997	30.4590	27.7995	25.4661	23.4124	19.9931	17.2944	15.1380	13.3941
42	34.1581	30.9941	28.2348	25.8206	23.7014	20.1856	17.4232	15.2245	13.4524
43	34.8100	31.5212	28.6616	26.1664	23.9819	20.3708	17.5459	15.3062	13.5070
44	35.4555	32.0406	29.0800	26.5038	24.2543	20.5488	17.6628	15.3832	13.5579
45	36.0945	32.5523	29.4902	26.8330	24.5187	20.7200	17.7741	15.4558	13.6055
46	36.7272	33.0565	29.8923	27.1542	24.7754	20.8847	17.8801	15.5244	13.6500
47	37.3537	33.5532	30.2866	27.4675	25.0247	21.0429	17.9810	15.5890	13.6916
48	37.9740	34.0426	30.6731	27.7732	25.2667	21.1951	18.0772	15.6500	13.7305
49	38.5881	34.5247	31.0521	28.0714	25.5017	21.3415	18.1687	15.7076	13.7668
50	39.1961	34.9997	31.4236	28.3623	25.7298	21.4822	18.2559	15.7619	13.8007

TABLE VIII.—AMERICAN EXPERIENCE TABLE OF MORTALITY

Age	Number living	Number dying	Age	Number living	Number dying	Age	Number living	Number dying
10	100,000	749	40	78,106	765	70	38,569	2,391
11	99,251	746	41	77,341	774	71	36,178	2,448
12	98,505	743	42	76,567	785	72	33,730	2,487
13	97,762	740	43	75,782	797	73	31,243	2,505
14	97,022	737	44	74,985	812	74	28,738	2,501
15	96,285	735	45	74,173	828	75	26,237	2,476
16	95,550	732	46	73,345	848	76	23,761	2,431
17	94,818	729	47	72,497	870	77	21,330	2,369
18	94,089	727	48	71,627	896	78	18,961	2,291
19	93,362	725	49	70,731	927	79	16,670	2,196
20	92,637	723	50	69,804	962	80	14,474	2,091
21	91,914	722	51	68,842	1,001	81	12,383	1,964
22	91,192	721	52	67,841	1,044	82	10,419	1,816
23	90,471	720	53	66,797	1,091	83	8,603	1,648
24	89,751	719	54	65,706	1,143	84	6,955	1,470
25	89,032	718	55	64,563	1,199	85	5,485	1,292
26	88,314	718	56	63,364	1,260	86	4,193	1,114
27	87,596	718	57	62,104	1,325	87	3,079	933
28	86,878	718	58	60,779	1,394	88	2,146	744
29	86,160	719	59	59,385	1,468	89	1,402	555
30	85,441	720	60	57,917	1,546	90	847	385
31	84,721	721	61	56,371	1,628	91	462	246
32	84,000	723	62	54,743	1,713	92	216	137
33	83,277	726	63	53,030	1,800	93	79	58
34	82,551	729	64	51,230	1,889	94	21	18
35	81,822	732	65	49,341	1,980	95	3	3
36	81,090	737	66	47,361	2,070			
37	80,353	742	67	45,291	2,158			
38	79,611	749	68	43,133	2,243			
39	78,862	756	69	40,890	2,321			

Answers to problems

*Answers to problems**

Exercise 1, Page 5

1. All are rational except $\sqrt{5}$.
2. All are rational except $\sqrt{3}$.
3. $-6 < -1$.
5. $0 > -3$.
6. $-2 > -3$.
7. -16.
9. 60.
10. -24.
11. -3.
13. 3.
14. -6.
15. -5; -7; -6; -6.
17. -10; -6; 16; 4.
18. -8; -16; -48; -3.
19. -9; 9; 0; 0.
21. 5; 15; -50; -2.
22. -7; -7; 0; ruled out.
23. (*a*) 0; (*b*) ruled out; (*c*) 0.
25. $a = 0$ and $b \neq 0$.

Exercise 2, Pages 7–8

1. $4x + 5y + 6$.
2. $8a - 7$.
3. $3a - b - 2$.
5. $4x - 6y + 5$.
6. $-2a + 6b + 5$.
7. 125.
9. -32.
10. 1,000,000.
11. .0001.
13. $\frac{9}{49}$.
14. $\frac{8}{125}$.
15. a^9.
17. x^6.
18. a^6.
19. a^{12}.
21. $16x^4y^4$.
22. $27a^3b^3$.
23. $100x^6$.
25. $-\dfrac{a^7}{b^7}$.
26. $-\dfrac{32a^5}{b^{10}}$.
27. $8a^3b^6c^{15}$.
29. a^3.
30. a^7.
31. (*a*) 1; (*b*) x, y, xy, y^2, xy^2, 1; (*c*) 1.
33. (*a*) 3; (*b*) 5, x, $5x$, 1; y, 1; 1, 2, 3, 4, 6, 12; (*c*) 5, 1, 12.
34. (*a*) 1; (*b*) t, $(x + y)$, $t(x + y)$, 1; (*c*) 1.
35. (*a*) 24; (*b*) $24y^3$; (*c*) $3x$.

* Answers are given to all problems except those whose numbers are multiples of four.

Exercise 3, Pages 10–11

1. $30a^{10}$.
2. $2ab^2$.
3. $-14a^5b^7$.
5. $12a^2 - 6ab$.
6. $-6x^2 + 12xy$.
7. $-6a^3 + 8a^2b + 10a^3b^3$.
9. $3x^3 + 10x^2 + 13x + 10$.
10. $15x^2 + 38x + 24$.
11. $4x^3 - 35x^2 + 18$.
13. $15x^3 + 29x^2 - 24x + 4$.
14. $x^4 - 1$.
15. $x^4 - 7x^2 + 1$.
17. $5a^3c^6$.
18. $-3x^8y^6z^7$.
19. $4x^2 + 3x - 2$.
21. $2x - 3$.
22. $4x + 7$.
23. $x^2 + 7x + 8 + \dfrac{9}{x-2}$.
25. $3a^3 + a^2 - 2a - 5 + \dfrac{4}{2a-5}$.
26. $4x^3 - 6x^2 + 2x - 7 + \dfrac{8}{3x+1}$.
27. $7x - 3 + \dfrac{x+8}{x^2 - 3x + 1}$.
29. $-5x^2 + 6x - 7$.
30. $4x^2 - 6xy^5 + 9y^{10}$.
31. $x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4$.

Exercise 4, Pages 13–14

1. $x^2 - 9$.
2. $x^2 - 49$.
3. $25x^2 - 1$.
5. $4y^2 - 49$.
6. $16y^2 - 9$.
7. $81x^2 - 16y^2$.
9. $x^2 + 12x + 36$.
10. $x^2 + 10x + 25$.
11. $9x^2 + 12x + 4$.
13. $4t^2 - 28t + 49$.
14. $36r^2 - 12r + 1$.
15. $25x^2 - 90xy + 81y^2$.
17. $x^2 + 8x + 12$.
18. $x^2 + 9x + 20$.
19. $x^2 - 8x + 7$.
21. $x^2 + 7x - 18$.
22. $x^2 + 2x - 35$.
23. $x^2 - 2x - 3$.
25. $3x^2 + 10x + 8$.
26. $5x^2 + 22x + 21$.
27. $21y^2 + 13y + 2$.
29. $45x^2 - 23xy + 2y^2$.
30. $40x^2 - 31xy + 6y^2$.
31. $35x^2 - 58x + 24$.
33. $2x^2y^2 - axy - 21a^2$.
34. $2a^2b^2 + 3abt - 5t^2$.
35. $8x^2 + 18xy - 5y^2$.
37. $12x^2 + 23x - 77$.
38. $55x^2 - 3x - 18$.
39. $30r^2 + 31rs - 6s^2$.
41. $9x^2 - \frac{1}{16}$.
42. $x^2 - .04$.
43. $x^2 + 1.4x + .49$.
45. $64x^8 - 80x^4y^3 + 25y^6$.
46. $25x^2 - \dfrac{5x}{3} + \dfrac{1}{36}$.
47. $9r^2 + 36rs + 36s^2$.
50. $u^2 + 4uv + 4v^2 - 9w^2$.
51. $u^2 + v^2 + w^2 + 2uv + 2uw + 2vw$.
53. $r^2 + 2rs + s^2 - 5r - 5s + 6$.
54. $c^2 + 2cd + d^2 + c + d - 42$.
55. $r^2 + 2rs + s^2 - t^2 - 2tu - u^2$.

57. $49r^2 + 70rt + 25t^2 - s^2$.
58. $9u^2 + 24uv + 16v^2 - 33uw - 44vw + 30w^2$.
59. $r^2 + s^2 + t^2 + 2rs + 2rt + 2st$.

Exercise 5, Pages 16–17

1. $m(r + s)$.
2. $b(s - t)$.
3. $3x(4x - 1)$.
5. $-4xy(1 + 2y)$.
6. $7ab(3a - 1)$.
7. $x^4(x^4 + x^2 + 1)$.
9. $2a^2b^4(4a + 3b - 5a^6b^5)$.
10. $3x^2y^4(3y^3 - 4x^3y^2 + 2x)$.
11. $(x + 8)(x - 8)$.
13. $(5x + 6y)(5x - 6y)$.
14. $(9x + 4y)(9x - 4y)$.
15. $(y + \frac{1}{3})(y - \frac{1}{3})$.
17. $7(2y + z)(2y - z)$.
18. $3(c + 5d)(c - 5d)$.
19. $(4x + y^8)(4x - y^8)$.
21. $(x + 4)^2$.
22. $(x - 8)^2$.
23. $(x - 10y)^2$.
25. $(6x + 5y)^2$.
26. $(7x - 4y)^2$.
27. $(x^5 - 11)^2$.
29. $(t + 7)^2$.
30. $(w - 6)^2$.
31. $(3x - 2y)^2$.
33. $(x + 2)(x + 3)$.
34. $(x + y)(x + 16y)$.
35. $(x + 6)(x - 1)$.
37. $(x - 8)(x + 5)$.
38. $(t - 8)(t + 6)$.
39. $(x - 4y)(x - 9y)$.
41. $(2x + 1)(x + 1)$.
42. $(5x + 9)(x + 1)$.
43. $(3r + 8)(r - 1)$.
45. $(5x - 6y)(x + y)$.
46. $(2x + 1)(x - 1)$.
47. $(7x - 11)(x - 1)$.
49. $(2x - 5)(2x - 7)$.
50. $(3x + 5)(2x + 7)$.
51. $(2x + 3)(5x - 2)$.
53. $5x(x + 2y)(x - 2y)$.
54. $2a(x + 7)^2$.
55. $10x^3(x - 4)^2$.
57. $(x^2 + 4y^2)(x + 2y)(x - 2y)$.
59. $(x^2 + 3)(x + 3)(x - 3)$.
61. $(3x - 2)(2 - x)$.
62. $(2x + 7)(5 - x)$.
63. $(3t - 8)(2t + 7)$.
65. $(25x + 2)(3x - 4)$.
66. $(3x - 8)(6x - 5)$.
67. $-(x - 13y)^2$.
70. $(a + 5s + 5t)(a - 2s - 2t)$.
71. $(a + 1)(a + 5)(a + 3)^2$.
73. $(r + s + u + v)(r + s - u - v)$.
74. $(a - b + 2c - 2d)(a - b - 2c + 2d)$.
75. $(s + t + 6)^2$.

Exercise 6, Page 19

1. $(x + y)(5a + 6b)$.
2. $(x + 8y)(2r - 3s)$.
3. $(a + b)(x + y)$.
5. $(a - 2b)(5r - 6s)$.
6. $(4a + 7b)(z - 2w)$.
7. $(x - 1)(6x^2 + 7)$.
9. $(3x + 2)(7x^2 - 5)$.
10. $(x + 6)(x^2 + 9)$.
11. $(a - b)(w - z)$.
13. $(x + a - 3b)(x - a + 3b)$.
14. $(r + a + 5b)(r - a - 5b)$.
15. $(x + 9y + r)(x + 9y - r)$.
17. $(11x + a + 8b)(11x - a - 8b)$.
18. $(10x + c - 7d)(10x - c + 7d)$.
19. $(6r + y + 1)(6r - y - 1)$.
21. $(x^2 + 4x + 9)(x^2 - 4x + 9)$.
22. $(x^2 + 3x + 5)(x^2 - 3x + 5)$.
23. $(x^2 + 2x + 2)(x^2 - 2x + 2)$.

25. $(x^2 + 6x + 1)(x^2 - 6x + 1)$.
26. $(x^2 + 5xy + 10y^2)(x^2 - 5xy + 10y^2)$.
27. $(x^2 + 7xy - 3y^2)(x^2 - 7xy - 3y^2)$.
29. $(5x^2 + 6xy - 7y^2)(5x^2 - 6xy - 7y^2)$.
30. $(2x^2 + 3xy - 4y^2)(2x^2 - 3xy - 4y^2)$.
31. $(3x^2 + 8xy + 6y^2)(3x^2 - 8xy + 6y^2)$.
33. $(a + b)(a + b + 1)$.
34. $(5x + 4y)(5x - 4y - 1)$.
35. $(3r + 2s)(1 - 3r + 2s)$.
37. $(s - 3t)(a + b)(a - b)$.
38. $(7w - z)(x + 2y)(x - 2y)$.
39. $(a + 5b + x - 9y)(a + 5b - x + 9y)$.
41. $(2r + 3t + x + 4y)(2r + 3t - x - 4y)$.
42. $(6u - v + z - 5)(6u - v - z + 5)$.
43. $ab(x + y)(c + d - 1)$.
45. $(x + y + z)(3 - a + b)$.
46. $(2 - y)(y + 3)(y - 3)$.

Exercise 7, Page 21

1. $(x + 2)(x^2 - 2x + 4)$.
2. $(x - 2)(x^2 + 2x + 4)$.
3. $(r - s)(r^2 + rs + s^2)$.
5. $(x - 10)(x^2 + 10x + 100)$.
6. $(x + 10)(x^2 - 10x + 100)$.
7. $\left(s + \frac{1}{3}\right)\left(s^2 - \frac{s}{3} + \frac{1}{9}\right)$.
9. $(4x + 5)(16x^2 - 20x + 25)$.
10. $(3x - 4y)(9x^2 + 12xy + 16y^2)$.
11. $(2x^2 - 1)(4x^4 + 2x^2 + 1)$.
13. $(x - y)(x^2 + xy + y^2)(x^6 + x^3y^3 + y^6)$.
14. $(x + y)(x^2 - xy + y^2)(x^6 - x^3y^3 + y^6)$.
15. $(x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)(x^{10} - x^5y^5 + y^{10})$.
17. $(r + s + 3)(r^2 + 2rs + s^2 - 3r - 3s + 9)$.
18. $(p - q - 5)(p^2 - 2pq + q^2 + 5p - 5q + 25)$.
19. $(5x - u + v)(25x^2 + 5ux - 5vx + u^2 - 2uv + v^2)$.
21. $2x(x - 2)(x^2 + 2x + 4)$.
22. $3a^2x^2(3 + x)(9 - 3x + x^2)$.
23. $(x + y)(x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6)$.
25. $(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)$.
26. $(x + 2)(x^4 - 2x^3 + 4x^2 - 8x + 16)$.
27. $(x^4 + y^4)(x^2 + y^2)(x + y)(x - y)$.
29. $(x^2 + y^2)(x^4 - x^2y^2 + y^4)$.
30. $(x - y)(x + y)(x^2 + y^2)(x^2 + xy + y^2)(x^2 - xy + y^2)(x^4 - x^2y^2 + y^4)$.

Exercise 8, Pages 23–24

1. 450; 15.
2. 84; 2.
3. $90x^2y^3$; $3xy$.
5. $(x+1)(x+2)(x+3)(x+4)$; $x+3$.

6. $24x(x+3)(x-3)^2$; $2(x-3)$.
7. $(x-4)^2(x+9)$.
9. $12(x+7)$.
10. $(x+8)(x-8)(x+9)(x-9)$.
11. $3(x-4)(x+2)$.
13. $(x-5)^2(x-1)(x+3)$.
14. $8(3x-5)^2$.
15. $(x+y)(x-y)(x^2-xy+y^2)$.
17. $(x-y)^3(x+y)$.
18. $(x+1)(x-1)(x^2+x+1)(x^2-x+1)$.

Exercise 9, Pages 26–28

1. $\frac{7}{11}$. 2. $\frac{2}{7}$. 3. $\dfrac{3a^6b^5c^2}{4}$. 5. $\dfrac{3b^4}{2a}$.

6. $\dfrac{14a^6}{5b}$. 7. 3. 9. $\frac{4}{5}$. 10. $\dfrac{1}{2x-5}$.

11. $\dfrac{x-7}{x+7}$. 13. $\dfrac{x-8}{x+11}$. 14. $\dfrac{2(x-2)}{x-3}$. 15. $\dfrac{x}{x+y}$.

17. $\dfrac{1}{x^2-5x+25}$. 18. $\dfrac{x^2+2x+4}{x+2}$. 19. ac. 21. $b(x+y)$.

22. x^2-3x. 23. $2a^2-3ab$. 25. $4x+8$. 26. 7.

27. $-\dfrac{x}{7}$. 29. $\dfrac{a}{b}$. 30. $-\dfrac{ab}{c}$. 31. $-\dfrac{a+b}{6}$.

33. $-\dfrac{4a}{b+c}$. 34. $\dfrac{r+s}{x+y}$. 35. True. 37. True.

38. True. 39. True. 41. False. 42. True.

43. False. 45. -3. 46. $\dfrac{a}{a-b}$. 47. $\dfrac{x-y}{x+y}$.

49. $-\dfrac{a^2+ab+b^2}{x}$. 50. $\dfrac{x^2+y^2}{(x^2+xy+y^2)(x^2-xy+y^2)}$.

51. $a+b+1$. 53. (a), (b), (c), (e).

Exercise 10, Pages 30–32

1. $\dfrac{4x+5y-6}{7}$. 2. 4. 3. $\dfrac{2x+3}{6}$.

5. 1. 6. $\dfrac{3x}{x+1}$. 7. $\frac{19}{24}$.

9. $-\frac{68}{35}$. 10. $\dfrac{x-5}{12}$. 11. $-\dfrac{14a+33}{9}$.

13. $\dfrac{3x-2}{6}$. 14. $\dfrac{2b-1}{ab}$. 15. $\dfrac{y-x^3}{x^2y}$.

17. 5.

18. $-\dfrac{2}{(x+1)(x+3)}$.

19. $\dfrac{17x+1}{24(x+5)}$.

21. $\dfrac{3x+70}{(x-7)(x+6)}$.

22. $\dfrac{6a-1}{3a-1}$.

23. 4.

25. $\dfrac{15x^2-76x+8}{12(x+2)(x-2)}$.

26. $\dfrac{1}{x-3}$.

27. 0.

29. $\dfrac{x+4}{x}$.

30. $\dfrac{2x^2+12x-5}{(x+2)(x+5)(x+6)}$.

31. $\dfrac{5}{x-5}$.

33. $\dfrac{x^2+28}{(x+5)(x-2)(x-5)}$.

34. $\dfrac{x^2+2x-20}{(x+1)(x+3)(x+4)}$.

35. $\dfrac{1}{x+3}$.

37. $\dfrac{5x^2+1}{6(x+1)^2}$.

38. $-\dfrac{(x-2)(x-6)}{(x-4)^2}$.

39. $\dfrac{x+2}{(x+3)^2}$.

41. $\dfrac{6}{(x-1)(x-3)^2}$.

42. $\dfrac{3}{x-3}$.

43. $\dfrac{2(x+3)}{(x-3)^2}$.

45. $-\dfrac{5x+21}{8x^3+125}$.

46. $\dfrac{5x^2+8x+4}{(5x-2)^3}$.

47. $\dfrac{5x^3-18x-9}{(x+1)^2(x-1)}$.

Exercise 11, Pages 33–34

1. $\frac{1}{2}$.

2. $\frac{15}{7}$.

3. $\dfrac{5bx}{3}$.

5. $\dfrac{7}{a^2b}$.

6. $\frac{1}{3}$.

7. $\dfrac{3b^2}{2ax^3}$.

9. $\dfrac{4a^2}{5x}$.

10. $\dfrac{25x^6y}{84a^3}$.

11. -1.

13. $\frac{1}{7}$.

14. a.

15. $\dfrac{3}{x}$.

17. $\dfrac{x+1}{x-2}$.

18. $\dfrac{x+1}{x+2}$.

19. $\dfrac{x-4}{2(x-9)^2}$.

21. 1.

22. $\dfrac{1}{x+1}$.

23. x.

25. $x^2-4x+16$.

26. $y-x$.

27. (*a*) $-\dfrac{1}{3}$; (*b*) $\dfrac{s}{r}$; (*c*) t; (*d*) $\dfrac{1}{m+n}$; (*e*) $\dfrac{x+y}{2x}$.

29. 1.

Exercise 12, Pages 35–37

1. $\frac{20}{57}$. 2. $\frac{7}{72}$. 3. $\frac{48}{41}$. 5. $\frac{8}{35}$.
6. 2. 7. s. 9. $\frac{x(x-3)}{x+2}$. 10. $\frac{b}{b-8}$.
11. $\frac{4x-3y}{9}$. 13. $\frac{x^2+1}{2}$. 14. $\frac{1}{a(a-3)}$. 15. $7-x$.
17. $\frac{x+2}{x+6}$. 18. $\frac{7(x+1)}{3(x+2)}$. 19. $\frac{(x+1)(x+2)}{(x+4)(x+5)}$.
21. $\frac{x+1}{x+3}$. 22. $\frac{x-5}{x-7}$. 23. $\frac{x-2}{x-1}$.
25. $\frac{2}{3(x-5)}$. 26. $\frac{x-3}{(x+2)(x+3)}$. 27. $\frac{x-6}{11x-18}$.
29. $\frac{2}{7}$. 30. $\frac{1}{(x-4)^2}$. 31. $-\frac{1}{4}$.

Exercise 13, Page 40

1. Identity. 2. Conditional. 3. Conditional.
5. Yes. 6. Yes. 7. Yes.

Exercise 14, Pages 44–47

1. $\frac{2}{3}$. 2. -9. 3. $-\frac{1}{2}$. 5. 0.
6. $-\frac{5}{3}$. 7. 16. 9. $\frac{5}{4}$. 10. 18.
11. $\frac{3}{10}$. 13. $\frac{4}{3}$. 14. 0. 15. .13.
17. -2.4. 18. $\frac{1}{3}$. 19. 6. 21. 11.
22. $\frac{1}{2}$. 23. 5. 25. $-\frac{1}{2}$. 26. $-\frac{2}{3}$.
27. $\frac{1}{10}$. 29. $-\frac{2}{3}$. 30. 11. 31. -2.
33. $\frac{1}{3}$. 34. 0. 35. $-\frac{2}{5}$. 37. 0.
38. -3. 39. 10. 41. $-\frac{1}{2}$. 42. -4.
43. -2. 45. No solution. 46. No solution. 47. No solution.
49. $\frac{11}{2}$. 50. 1. 51. $\frac{a-b}{7}$. 53. $\frac{6}{a+1}$.
54. $\frac{b}{a-1}$. 55. $\frac{a}{5-b}$. 57. $a+1$. 58. $2a$.
59. $\frac{bc}{a}$. 61. ab. 62. $\frac{b}{a}$. 63. $\frac{b-2c}{a-c}$.
65. $\frac{ab-9}{3a-1}$. 66. $\frac{b-a}{1+ab}$. 67. $\frac{bc-a+b}{c}$. 69. $\frac{2ab}{3a+4b}$.
70. $a+b$. 71. $\frac{2s}{t^2}$. 73. $\frac{5}{9}(F-32)$. 74. $\frac{S-a}{S-l}$.

75. $\dfrac{7-6y}{4+5y}$. **77.** $\dfrac{3(x+2)}{5-x}$. **78.** $-2x$.

79. (1) $\dfrac{A}{1+rt}$; (2) $\dfrac{A-P}{Pr}$. **81.** (1) $\dfrac{2S}{a+l}$; (2) $\dfrac{2S-nl}{n}$.

82. (1) $\dfrac{2A-hB}{h}$; (2) $\dfrac{2A}{b+B}$. **83.** (1) $\dfrac{ab}{a+b}$; (2) $\dfrac{bf}{b-f}$.

Exercise 15, Pages 49–51

1. 21; 22.
2. 34; 35; 36; 37.
3. 13.
5. 34 nickels; 26 dimes.
6. 58 earn $40; 12 earn $45.
7. $180; $90; $45.
9. 11 nickels; 22 dimes; 66 quarters.
10. 21 ft.; 28 ft.; 35 ft.
11. 30 lb. of 80¢ tea; 70 lb. of 50¢ tea.
13. 10 oz. **14.** 35 lb. **15.** $2\frac{7}{9}$ lb. **17.** 6 quarts.
18. $\dfrac{b}{16}$ gal. of pure alcohol; $\dfrac{15b}{16}$ gal. of 20% solution.
19. $4\frac{2}{3}$ hr.
21. $18\frac{3}{4}$ mi.
22. 120 mi.
23. 4 mph.
25. 170 mph.
26. $\dfrac{3ab}{3a+2}$ mph.
27. $6\frac{9}{16}$ ft. from boy.
29. 22 years from now.
30. 16 years and 48 years.
31. $2\frac{2}{5}$ hr.
33. $850 at 3%; $150 at 5%.
34. $13,200.
35. $4200.
37. $3 : 16\frac{4}{11}$.
38. $1 : 38\frac{2}{11}$.
39. $4 : 26\frac{122}{143}$.

Exercise 16, Pages 54–55

1. s equals the f-function of t.
6. the cost per gallon.
11. $D = .70x$.
13. $V = 60 + 5n$.
14. $I = .03x$.
15. $F = 4 + .03(x - 100)$.
17. 17.
18. -48.
19. 2.
21. 20.
22. $3a^2 - 4a$.
23. $\frac{5}{3}$.
25. $25w^2 + 20w + 4$.
26. $5w^2 + 2$.
27. $\frac{12}{17}$.
29. $20r + 8$.
30. $15z^2 - 20z + 2$.
31. $3; \dfrac{27}{125}; \dfrac{a-6b}{2(a-5b)}$.
33. (a) $y = \pi x - 2x + 7$; (b) $x = \dfrac{7-y}{2-\pi}$.
34. 20; 4; -16.

Exercise 17, Pages 57–59

9. (*a*) II; (*b*) III; (*c*) IV; (*d*) I; (*e*) IV.

Exercise 19, Page 65

19. (*a*) $y = \dfrac{8 - 3x}{7}$; (*b*) $x = \dfrac{8 - 7y}{3}$.

20. (*a*) $y = \dfrac{4x - 7}{5}$; (*b*) $x = \dfrac{5y + 7}{4}$.

Exercise 20, Pages 67–68

1. $(x = -2.1, y = 2.9)$.
2. $(x = 3.6, y = .8)$.
3. $(x = 3, y = -1.6)$.
5. $(x = -1.2, y = -4.2)$.
6. $(x = 1.2, y = -1.3)$.
7. $(x = 1.8, y = 2.5)$.
9. $(x = 3.2, y = -1.6)$.
10. $(x = -4, y = 1.4)$.
11. No solution. Inconsistent.
13. Infinitely many solutions. Dependent. $(x = 0, y = 4)$; $(x = 6, y = 0)$.
14. No solution. Inconsistent.
15. No solution. Inconsistent.

Exercise 21, Pages 71–72

1. $(x = -2, y = 3)$.
2. $(x = 1, y = 6)$.
3. $(x = -1, y = -2)$.
5. $(x = \frac{5}{2}, y = 0)$.
6. $(x = \frac{7}{8}, y = \frac{1}{8})$.
7. $(x = 9, y = 1)$.
9. $(x = -\frac{5}{8}, y = -\frac{3}{4})$.
10. $(x = 0, y = 0)$.
11. $(x = \frac{6}{7}, y = -\frac{1}{7})$.
13. $(x = 7, y = -\frac{3}{2})$.
14. $(x = -\frac{1}{10}, y = \frac{13}{10})$.
15. $(x = -10, y = 9)$.
17. $(x = 0, y = -\frac{2}{3})$.
18. $(x = \frac{1}{2}, y = \frac{1}{2})$.
19. $(x = 2, y = 3)$.
21. $(x = 0, y = 0)$.
22. $(x = 1, y = 1)$.
23. $(x = \frac{5}{4}, y = -\frac{3}{8})$.
25. Dependent.
26. Inconsistent.
27. Inconsistent.
29. Inconsistent.
30. Dependent.
31. $(x = \frac{3}{2}, y = \frac{1}{3})$.
33. $(x = \frac{1}{6}, y = \frac{1}{2})$.
34. $(x = -2, y = \frac{1}{13})$.
35. $(x = a^2 + b^2, y = -ab)$.
37. $(x = a, y = b)$.
38. $\left(x = \dfrac{ab}{c}, y = d\right)$.

Exercise 22, Page 73

1. $(x = 3, y = 2, z = 0)$.
2. $(x = 1, y = 1, z = 2)$.
3. $(x = \frac{1}{2}, y = 5, z = \frac{3}{2})$.
5. $(x = 3, y = 1, z = \frac{1}{2})$.
6. $(x = 10, y = 11, z = -12)$.
7. $(x = \frac{1}{2}, y = 1, z = \frac{3}{2})$.
9. $(x = -\frac{1}{6}, y = -\frac{1}{3}, z = -\frac{1}{2})$.
10. $(x = -1, y = 3, z = -1)$.
11. $(x = 2, y = 0, z = 0)$.
13. $(x = \frac{1}{3}, y = \frac{2}{3}, z = 5)$.
14. $(x = 2, y = 3, z = 4)$.
15. $(a = 2, b = 0, c = 1, d = -6)$.

Exercise 23, Pages 75–77

1. 18 cents; 68 cents.
2. 6 free throws; 21 field goals.
3. \$4.06.
5. 90 mph; 10 mph.
6. $\frac{1}{2}$ mph; $2\frac{1}{2}$ mph.
7. \$5 and \$7.
9. Woman, 120 lb.; boy, 30 lb.
10. $\frac{5}{9}$.
11. A, 45 hours; B, 36 hours.
13. 18 ft. per sec.; 12 ft. per sec.
14. $(m = 2, b = -1)$.
15. Stocks, \$2500; bonds, \$6000; savings, \$1500.
17. Plane, $4\frac{1}{5}$ hours; train, $2\frac{1}{5}$ hours; auto, $\frac{3}{5}$ hour.
18. 64 and 12.
19. 761.

Exercise 24, Page 79

1. $(x = -5, y = \frac{11}{2})$.
2. $(x = 9, y = 6)$.
3. $(x = \frac{5}{2}, y = 0)$.
5. $(x = 16, y = -14)$.
6. $(x = -\frac{23}{6}, y = -\frac{11}{4})$.

Exercise 25, Page 82

1. $(x = 1, y = 2, z = 2)$.
2. $(x = 1, y = -2, z = 3)$.
3. $(x = \frac{1}{2}, y = \frac{1}{4}, z = \frac{1}{8})$.
5. $(x = -\frac{5}{12}, y = \frac{7}{12}, z = \frac{1}{12})$.
6. $(x = \frac{1}{6}, y = \frac{5}{6}, z = 1)$.

Exercise 26, Pages 84–85

1. -9.
2. 9.
3. -8.
5. 32.
6. 8.
7. $\frac{9}{25}$.
9. a^5.
10. a^{4+k}.
11. t^8.
13. a^6.
14. x^{28}.
15. r^2s^{10}.
17. $81x^4y^{12}$.
18. $.09w^4z^{18}$.
19. $-125c^9d^{18}$.
21. a^4.
22. x^8.
23. $\frac{1}{t^4}$.
25. $\frac{y^2}{x}$.
26. $\frac{r^2}{s^4}$.
27. 1000.
29. $\frac{x^6}{y^{10}}$.
30. $\frac{64}{c^{12}}$.
31. $\frac{125r^3s^6}{t^{12}}$.
33. $\frac{x^{12}y^{16}}{16z^4}$.
34. $\frac{a^{pn}b^{2n}}{c^{n^2}}$.
35. $\frac{4}{5y^{10}}$.
37. 250.
38. -12.

Exercise 27, Pages 86–87

1. ± 11.	2. ± 1.	3. $\pm \frac{2}{5}$.	5. .6.
6. $\frac{1}{8}$.	7. 9.	9. -5.	10. $-\frac{1}{2}$.
11. $\frac{2}{3}$.	13. 12.	14. 10.	15. -3.
17. $\frac{1}{2}$.	18. -1.	19. .2.	21. $-x$.
22. $-t$.	23. $4y^8$.	25. 1865.	26. -1492.
27. -1776.	29. 1066.	30. -1620.	31. $-7ab^2$.

Exercise 28, Pages 90–92

1. 7.	2. 2.	3. $\frac{1}{2}$.	5. 9.
6. 8.	7. 1000.	9. $\frac{1}{32}$.	10. .0625.
11. $\frac{2}{3}$.	13. 8.	14. 1.	15. 4.
17. 3.	18. 4.	19. $\frac{1}{7}$.	21. $\frac{1}{81}$.
22. $-\frac{1}{64}$.	23. 8.	25. $\frac{1}{8}$.	26. $\frac{1}{100}$.
27. $\frac{1}{9}$.	29. 8.	30. 1000.	31. $\frac{2}{3}$.
33. 25.	34. 50,000.	35. $\frac{1}{x^3}$.	37. $\frac{4}{a}$.
38. $\frac{6}{y^2}$.	39. a^2b^6.	41. $\frac{t}{rs^2}$.	42. $\frac{c^8}{ab^7}$.
43. $80x^3$.	45. $5x^{-3}$.	46. $(1.02)^{-3}W^2$.	47. $7a^{-1}b^{-3}x$.
49. $\sqrt[7]{a^2}$.	50. $\sqrt[5]{x^3}$.	51. $\sqrt[3]{100x^2}$.	53. $5\sqrt{a}$.
54. $b\sqrt[5]{x^4}$.	55. $a^{\frac{5}{7}}$.	57. $(x+y)^{\frac{1}{2}}$.	58. $3(a-c)^{\frac{1}{4}}$.
59. $x^{\frac{4}{3}}$.	61. 2.	62. 1000.	63. $x^{\frac{4}{5}}$.
65. $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{3}}}$.	66. $\frac{y^{\frac{3}{4}}}{x^{\frac{3}{5}}}$.	67. $4x^{\frac{1}{8}}$.	69. x^{18}.
70. y^{15}.	71. $\frac{9}{x^{10}}$.	73. $2b^2z$.	74. $16x^8y^{20}$.
75. $\frac{1}{a^{15}b^{20}}$.	77. $\frac{a^{12}}{8b^3}$.	78. $\frac{y^{32}}{625x^4}$.	79. $\frac{a^{10}}{32}$.
81. $\frac{x^4}{25}$.	82. $\frac{a^8}{16}$.	83. $\frac{3y^6}{4x^4}$.	85. $\frac{4b^3}{7a^4}$.
86. $\frac{a^4}{4b^5}$.	87. 4.	89. $\frac{63b}{b+45a^2}$.	90. $\frac{6a^4}{12+a^4}$.
91. $\frac{7a^2}{b^2(1-a^3)}$.	93. $\frac{8x^2-1}{4x}$.	94. $\frac{x+1}{x^{\frac{3}{2}}}$.	95. $\frac{49a^{\frac{1}{2}}}{81b^4}$.
97. $\frac{8x^3}{y^{21}}$.	98. $\frac{9a^4}{64b^{10}}$.	99. $\frac{4y^4}{9x^6}$.	101. $\frac{6ab^8}{5}$.
102. $\frac{8x^6}{27y^9}$.		103. $\frac{7^na^{2n}}{b^{n^2}}$.	

105. $e^{2x} + 2 + e^{-2x}$.
106. $25x^2 - y^{-6}$.
107. $a - 5a^{\frac{1}{2}}b^{\frac{1}{2}} + 6b$.
109. $x - y$.
110. $a + 8b$.
111. $2x^{-2} + 4x^{-1} + 5$.
113. $x^{\frac{3}{4}} - x^{\frac{1}{4}} - x^{-\frac{1}{4}}$.
114. $8x^{\frac{3}{2}} + 9x + 1$.
115. (*a*) 2^{n+1}; (*b*) 2^{2n+3}; (*c*) 2^{2n+1}.
118. $9(10)^{-9}$.
119. $4(10^{-11})$.
121. $25(10^{12})$.
122. 260,000,000,000.
123. 67,000,000,000,000.
125. 0.000 000 000 000 016.

Exercise 29, Pages 94–95

1. $\sqrt{3}$.
2. $\sqrt[3]{5}$.
3. $\sqrt[5]{x^3}$.
5. $\sqrt{10x}$.
6. $\sqrt[4]{9y^3}$.
7. $\sqrt[4]{x+1}$.
9. $2\sqrt{5}$.
10. $3\sqrt{2}$.
11. $4y^8$.
13. $2\sqrt[3]{7}$.
14. $3y^9$.
15. $-3x^2\sqrt[3]{2x}$.
17. $2xy\sqrt[4]{3xy^3}$.
18. $2xy^2\sqrt[4]{5x^2y}$.
19. $(x+y)\sqrt{x-y}$.
21. $2\sqrt{x^2+4}$.
22. $2\sqrt[3]{x^3+8}$.
23. $\dfrac{rs}{2t^2}\sqrt[3]{rs^2}$.
25. $\dfrac{\sqrt{6}}{3}$.
26. $\dfrac{\sqrt{30}}{10}$.
27. $\dfrac{\sqrt{35x}}{5x}$.
29. $\dfrac{\sqrt[3]{3axy^2}}{3xy}$.
30. $\dfrac{\sqrt[3]{98ab^2c}}{7bc}$.
31. $\dfrac{\sqrt[4]{4x^3y^2z}}{xyz}$.
33. $\dfrac{2\sqrt{a}}{3b}$.
34. $\dfrac{2x^3\sqrt{10}}{y^4}$.
35. $\dfrac{7x^2\sqrt{2}}{2}$.
37. $\dfrac{2x^{32}\sqrt{11y}}{y^4}$.
38. $\dfrac{3s^4\sqrt{6t}}{t^2}$.
39. $-\dfrac{5\sqrt[3]{a^2b}}{ab^2}$.
41. $\dfrac{a^2\sqrt[3]{3a^2}}{3b^2}$.
42. $\dfrac{ab\sqrt[3]{25a}}{5}$.
43. $\dfrac{xy\sqrt[4]{2xy^3z^3}}{2z}$.
45. $\dfrac{2x\sqrt{35xz}}{15y^2z^3}$.
46. $\dfrac{3x^3\sqrt{6xy}}{8y^2z^3}$.
47. $\dfrac{2\sqrt[3]{a^3b^3+1}}{b}$.
49. $x\sqrt[3]{10x^2}$.
50. $x\sqrt{11x}$.
51. $\dfrac{\sqrt{5}}{4}$.
53. $\dfrac{3\sqrt{x(x+y^2)}}{x+y^2}$.
54. $\dfrac{(x+y)\sqrt{a}}{xy}$.
55. $(a+b)\sqrt[3]{a-b}$.
57. $a^4b^2\sqrt[n]{b}$.
58. $x^3y\sqrt[n]{y^2}$.
59. $\dfrac{\sqrt{(16-25a)a}}{5a}$.
61. $\dfrac{2b^9\sqrt[3]{11a^2}}{a^3}$.
62. $\dfrac{4z^3\sqrt{yz}}{5x^8y}$.
63. $\dfrac{4a^{24}\sqrt{231a}}{105b^{10}}$.

65. $3\sqrt{70} = 25.101.$ 66. $5\sqrt{7} = 13.230.$ 67. $\frac{8}{7}\sqrt{7} = 3.024.$
69. $\frac{\sqrt[3]{70}}{10} = .412.$ 70. $\frac{\sqrt[3]{99}}{6} = .771.$ 71. $\sqrt[3]{5} = 1.710.$
73. $4\sqrt{39}.$

Exercise 30, Pages 96–97

1. $9\sqrt{x}.$ 2. $3\sqrt{7}.$ 3. $8\sqrt{2}.$
5. $2\sqrt{3}.$ 6. $-7\sqrt{5}.$ 7. $\frac{2}{3}\sqrt{3}.$
9. $7\sqrt[3]{2}.$ 10. $\frac{2}{3}\sqrt[3]{9x}.$ 11. $\left(\frac{a+b}{5}\right)\sqrt[3]{25}.$
13. 0. 14. $\frac{17}{3}\sqrt{3} - \frac{15}{2}\sqrt{2}.$
15. $(ab^2 + bc^2 + a^2c)\sqrt[5]{a^2bc^3}.$ 17. $\frac{1}{6}\sqrt{6} + \frac{7}{2}\sqrt[3]{4}.$
18. $\frac{17}{4}\sqrt{2} - \frac{11x}{2}\sqrt[3]{x}.$ 19. $9\sqrt{5} + 4\sqrt[3]{5}.$

Exercise 31, Pages 97–98

1. $5\sqrt{2}.$ 2. $10x\sqrt{11}.$ 3. $2x\sqrt[3]{7x}.$
5. $\frac{\sqrt[3]{9}}{6}.$ 6. $\frac{2\sqrt{5}}{5}.$ 7. $125ab\sqrt{ab}.$
9. $49a^{14}x\sqrt[3]{xy^2}.$ 10. $80a^6x\sqrt{10x}.$ 11. $abc\sqrt[4]{ab^2c^3}.$
13. $x - 2\sqrt{xy} + y.$ 14. $79 - 24\sqrt{7}.$ 15. $56 + 20\sqrt{3}.$
17. 2. 18. $17 - 7\sqrt{5}.$
19. $\sqrt{55} - \sqrt{33} - \sqrt{35} + \sqrt{21}.$ 21. $40 - 5\sqrt{2} + 8\sqrt{3} - \sqrt{6}.$
22. $22 + 36\sqrt{3}.$ 23. 120.
25. $x^2 - 14x + 47.$ 26. $x^2 + 3x + 1.$
27. $1 + 3\sqrt{2}.$

Exercise 32, Pages 99–100

1. $\sqrt{5}.$ 2. $\sqrt{2}.$ 3. $\frac{\sqrt{2}}{2}.$
5. $2\sqrt{3}.$ 6. $2\sqrt{5}.$ 7. $\sqrt{2}.$
9. $\frac{\sqrt{3}}{x^2}.$ 10. $-\frac{\sqrt[3]{12}}{2}.$ 11. $x\sqrt[3]{7x}.$
13. $\frac{2\sqrt[3]{25}}{5}.$ 14. $\frac{\sqrt[5]{a^3x^4}}{x}.$ 15. $\frac{2\sqrt{3} - \sqrt{7}}{5}.$
17. $11 + 2\sqrt{35}.$ 18. $5 - 2\sqrt{3}.$ 19. $\sqrt{3} + \frac{\sqrt{10}}{2}.$

21. $12 + 5\sqrt{6} - 6\sqrt{2} - 5\sqrt{3}$. 22. $\dfrac{\sqrt{2} + \sqrt{6} - 2}{4}$.

23. .707. 25. 7.318. 26. .158. 27. .097.

Exercise 33, Page 101

1. $\sqrt[10]{32}, \sqrt[10]{36}; \sqrt[5]{6}$. 2. $\sqrt[6]{343}, \sqrt[6]{361}; \sqrt[3]{19}$. 3. $\sqrt[12]{27}, \sqrt[12]{25}; \sqrt[4]{3}$.
5. $\sqrt{a}$. 6. $\sqrt[20]{a^9}$. 7. $\sqrt[12]{128}$. 9. $\sqrt[6]{8x^2}$.
10. $\sqrt[12]{125a^{10}}$. 11. $\sqrt[10]{18}$. 13. $x\sqrt[12]{a^5x^7}$. 14. $ab\sqrt[12]{a^2b}$.
15. $\sqrt[21]{a}$. 17. $\dfrac{\sqrt[10]{9x^5}}{x}$. 18. $\dfrac{\sqrt[8]{64x}}{2}$. 19. $\dfrac{\sqrt[6]{81000}}{10}$.
21. $\sqrt[3]{5}$. 22. $\sqrt[6]{x}$. 23. $3\sqrt[4]{2a}$. 25. $2x\sqrt{x}$.
26. $4x\sqrt{5x}$. 27. $\sqrt{700x^3}$. 29. $\sqrt[3]{250a^8}$. 30. $\sqrt{\dfrac{9x^3}{2y}}$.
31. $\sqrt[5]{3x^5 - x^7}$. 33. 7.746.

Exercise 34, Pages 103–104

1. $6i$. 2. $2i$. 3. $\frac{5}{7}i$. 5. $i\sqrt{11}$.
6. $i\sqrt{5}$. 7. $2i\sqrt{3}$. 9. $\pm 11i$. 10. $\pm \frac{3}{4}i$.
11. $\pm \dfrac{i\sqrt{5}}{6}$. 13. $3 + 2i$. 14. $1 + 6i$. 15. $39 - 80i$.
17. $29 + 3i$. 18. $-8 + 44i$. 19. 34. 21. $-i$.
22. i. 23. -1. 25. 0. 26. 0.
31. $\frac{1}{7} - 4i$; $a = \frac{1}{7}$, $b = -4$; imaginary.
33. $-44 + 0 \cdot i$; $a = -44$, $b = 0$; real, rational.
34. $0 + 2\sqrt{7} \cdot i$; $a = 0$, $b = 2\sqrt{7}$; pure imaginary.

Exercise 35, Pages 104–105

1. $\frac{27}{64}$. 2. 2. 3. -16. 5. $10x^{50}$.
6. $81x^{36}$. 7. $\frac{1}{216}$. 9. 4. 10. 5.
11. $\dfrac{8}{x^9}$. 13. 4. 14. 320. 15. 7.
17. $\frac{31}{32}$. 18. -32. 19. $\frac{16}{81}$. 21. $\frac{25}{144}$.
22. 4. 23. $\dfrac{35y^3}{y^3 + 63x}$. 25. $\dfrac{5(x^2 - 5)}{x^2}$. 26. $\dfrac{x^2}{x^2 + 9x + 1}$.
27. $\dfrac{\sqrt{21}}{9}$. 29. $\sqrt[4]{2}$. 30. $\dfrac{\sqrt[3]{36}}{2}$. 31. $\dfrac{\sqrt{6}}{6}$.
33. $\dfrac{\sqrt[3]{22x^2y}}{x}$. 34. $\dfrac{b}{2a^2}$. 35. $3x\sqrt{7y}$. 37. $\dfrac{3\sqrt{2}}{10}$.

38. $\frac{3\sqrt{5}}{5}$. **39.** $x^2\sqrt[3]{11x^2}$. **41.** $\frac{15+2\sqrt{2}}{7}$. **42.** $\frac{37-11\sqrt{7}}{18}$.

43. $127 - 12\sqrt{77}$. **45.** $\frac{2x^6\sqrt[3]{4x}}{y^{16}}$. **46.** $\frac{2x^7\sqrt{13x}}{y^8}$.

47. $\frac{2x^6\sqrt[3]{12x^2}}{3}$. **49.** $x^3\sqrt[6]{2x^2}$. **50.** $x\sqrt[5]{3x^4}$.

51. $8r^6t^2\sqrt[4]{s^3t}$. **53.** $\frac{3\sqrt{4+x^{36}}}{x^{18}}$. **54.** $\frac{2\sqrt[3]{x^6+8}}{x^2}$.

55. $\sqrt[12]{4x^9}$. **57.** $x^2\sqrt[9]{2y}$. **58.** $e^x + e^{-x}$.

Exercise 36, Page 108

1. $\pm\sqrt{11}$. **2.** $\pm\sqrt{2}$. **3.** $\pm 4i$.

5. $\pm i\sqrt{6}$. **6.** $\pm\frac{3}{2}i$. **7.** $\pm\frac{2\sqrt{3}}{3}$.

9. $\pm\frac{2}{3}$. **10.** $\pm\frac{3}{4}$. **11.** $\pm\frac{\sqrt{r^2-s^2}}{4r}$.

13. $6; -1$. **14.** $2; 3$. **15.** $-7; -4$.
17. $\frac{1}{5}; 1$. **18.** $-\frac{1}{3}; -2$. **19.** $\frac{2}{7}; -1$.
21. $-\frac{10}{3}; -\frac{10}{3}$. **22.** $7; -\frac{3}{5}$. **23.** $\frac{1}{3}; 5$.
25. $0; \frac{5}{4}$. **26.** $0; \frac{2}{3}$. **27.** $0; -\frac{3}{8}$.
29. -2. **30.** -2. **31.** $-15b; -3b$.

Exercise 37, Page 110

1. $5 \pm \sqrt{3}$; $[3.268; 6.732]$. **2.** $-3 \pm \sqrt{10}$; $[-6.162; .162]$.
3. $-1 \pm \sqrt{7}$; $[-3.646; 1.646]$. **5.** $4 \pm 3i$.
6. $2 \pm i$. **7.** $36; -24$.
9. $\frac{5}{3}; -\frac{7}{3}$. **10.** $\frac{6}{5}; -\frac{2}{5}$.
11. $\frac{3\pm\sqrt{13}}{2}$; $[-.303; 3.303]$. **13.** $\frac{2\pm\sqrt{7}}{3}$; $[-.215; 1.549]$.
14. $1 \pm \frac{\sqrt{6}}{2}$; $[-.224; 2.224]$. **15.** $\frac{2\pm 10i}{3}$.
17. $\frac{5\pm\sqrt{17}}{4}$; $[.219; 2.281]$. **18.** $\frac{-3\pm\sqrt{89}}{10}$; $[.643; -1.243]$.
19. $\frac{-3\pm\sqrt{14}}{2}$; $[-3.371; .371]$.
21. (*a*) $y = \pm\sqrt{x^2-2x-6}$; (*b*) $x = 1 \pm \sqrt{y^2+7}$.

22. (a) $y = 3 \pm \sqrt{4 - x^2}$; (b) $x = \pm\sqrt{6y - y^2 - 5}$.

23. $t = \dfrac{-v_0 \pm \sqrt{v_0^2 + 2gs}}{g}$.

Exercise 38, Page 112

1. $\dfrac{-3 \pm \sqrt{5}}{2}$.
2. $\dfrac{7 \pm \sqrt{33}}{2}$.
3. 12; 18.
5. $5 \pm 3i$.
6. $-4 \pm 5i$.
7. $\frac{1}{2}$; $-\frac{3}{4}$.
9. $\dfrac{5 \pm \sqrt{33}}{4}$.
10. $\dfrac{5 \pm \sqrt{37}}{6}$.
11. $\dfrac{1 \pm i\sqrt{3}}{2}$.
13. $\dfrac{1 \pm \sqrt{2}}{2}$.
14. $\dfrac{3 \pm \sqrt{11}}{2}$.
15. $\dfrac{2 \pm \sqrt{14}}{5}$.
17. $\dfrac{-1 \pm \sqrt{7}}{3}$.
18. $\dfrac{-2 \pm \sqrt{22}}{3}$.
19. $\dfrac{-7h \pm \sqrt{37h^2 + 12}}{6}$.
21. $y = \dfrac{1 - 5x \pm \sqrt{17x^2 - 26x + 29}}{2}$.
22. $x = \dfrac{1 - 3y \pm \sqrt{5y^2 - 14y + 17}}{2}$.

Exercise 39, Pages 113–116

1. ± 4.
2. $\pm 4i$.
3. 0; 16.
5. 2; -7.
6. 5; -2.
7. $7 \pm 4i$.
9. $\dfrac{2 \pm 2\sqrt{7}}{3}$.
10. $\frac{1}{2} \pm \sqrt{2}$.
11. .2; -1.8.
13. $\pm \dfrac{b}{a}\sqrt{a^2 - x^2}$.
14. (a) $\pm\sqrt{r^2 - x^2}$; (b) $\pm\sqrt{x^2 + y^2}$.
15. $\pm \dfrac{\sqrt{2gs}}{g}$.
17. -4.
18. 0; -4.
19. 0; $-\frac{1}{3}$.
21. $-\frac{1}{2}$.
22. -1; $-\frac{1}{2}$.
23. 16 ft. by 18 ft.
25. 2 ft.
26. 5 ft.
27. 9, 10, 11.
29. 100 mph.
30. 15 mph.
31. 40 mph.
33. 1 min.; $2\frac{3}{5}$ min.
34. 7 : 07+ P.M.
35. 13.
37. 12.
38. 15 in. by 30 in.
39. 2; 4.
41. 25.
42. 8 mph.

Exercise 40, Page 118

1. 11.
2. 6.
3. $-2\sqrt{7}$.

5. No solution.
6. No solution.
7. $0; 3\sqrt{2}$.
9. $\frac{7}{9}$.
10. $\frac{1}{4}$.
11. 25.
13. 3.
14. 0.
15. $\frac{5}{8}$.
17. 1; 5.
18. 3.
19. $0; \frac{40}{9}$.
21. No solution.
22. No solution.
23. 1; 9.
25. 1; 3.
26. 6.
27. 4.
29. $45 - 8\sqrt{31} = .46^-$.
30. 4.
31. ± 1000.
33. $11; -\frac{31}{3}$.
34. $\pm(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$.

Exercise 41, Pages 120–121

1. $\pm 4; \pm i$.
2. $\pm\sqrt{2}; \pm 2i$.
3. $\pm 3; \pm 3i$.
5. $\pm 1; \pm\frac{1}{3}$.
6. $\frac{1}{5}; -\frac{1}{2}$.
7. 16.
9. 8; 27.
10. 1; 32.
11. $\pm 1; 2; 4$.
13. $1; 5; 3 \pm \sqrt{2}$.
14. $-\frac{1}{3}; -\frac{11}{3}; -2 \pm \sqrt{5}$.
15. $5; -1; -3; -21$.
17. $\frac{1}{2}; -1; \frac{1 \pm i\sqrt{3}}{2}$.
18. $1; 5; -\frac{1}{2}; -\frac{5}{2}$.
19. $-1; -2; -5; -6$.
21. $4; -3; 1 \pm 3\sqrt{2}$.
22. $1; -9$.
23. $\pm 1; 3; 5$.
25. 0; 4; 6.
26. $0; 2; \frac{1}{5}$.
27. $1; \frac{-1 \pm i\sqrt{3}}{2}$.
29. $-\frac{4}{3}; \frac{2 \pm 2i\sqrt{3}}{3}$.
30. $\frac{5}{2}; \frac{-5 \pm 5i\sqrt{3}}{4}$.

Exercise 42, Pages 124–125

1. Real; unequal; rational.
2. Imaginary; unequal.
3. Real; equal; rational.
5. Imaginary; unequal.
6. Real; unequal; rational.
7. Real; unequal; irrational.
9. Real; unequal; irrational.
10. Real; unequal; rational.
11. $-\frac{10}{3}; \frac{8}{3}$.
13. $-\frac{7}{2}; -2$.
14. $\frac{7}{6}; \frac{4}{3}$.
15. $0; -\frac{11}{5}$.
17. $3; -4$.
18. $-3; -\frac{7}{2}$.
19. $x^2 - 16x + 63 = 0$.
21. $x^2 - 5x - 24 = 0$.
22. $x^2 - 10x + 25 = 0$.
23. $5x^2 - 18x - 8 = 0$.
25. $14x^2 + 13x + 3 = 0$.
26. $x^2 - 18 = 0$.
27. $x^2 - 10x + 19 = 0$.
29. $x^2 + 14x + 50 = 0$.
30. $9x^2 - 12x + 29 = 0$.
31. $2x^2 - 6x + 7 = 0$.
33. (*a*) 6; (*b*) -3; (*c*) $\frac{23}{4}$; (*d*) 5.
34. (*a*) 0, 4; (*b*) $\frac{9}{2}$; (*c*) 28; (*d*) $\frac{16}{3}$.
35. (*a*) -2; (*b*) 160; (*c*) 48; (*d*) $-\frac{10}{[illegible]}$

Exercise 43, Pages 129–130

13. $-2; 3$.
14. $-4; 2$.
15. $-4.5; 1.5$.
17. $-1.7; 1.2$.
18. $2; 4$.
19. Tangent to x-axis.
21. Does not meet x-axis.
22. Tangent to x-axis.
23. Cuts x-axis in two points.
25. Min. value of -4 when $x = -1$.
26. Max. value of $15\frac{1}{2}$ when $x = -\frac{5}{2}$.
27. Max. value of 19 when $x = 4$.
29. 10 and 10.
30. 10 and 10.
31. 25 ft. by 25 ft.
33. 40 ft. by 50 ft.
34. $60 per month.

Exercise 44, Pages 132–133

For abbreviation, (1, 2) is used to indicate $(x = 1, y = 2)$.

1. $(1, 2); (3, 0)$.
2. $(0, 3); (\frac{9}{5}, -\frac{3}{5})$.
3. $(3, 1); (-3, 4)$.
5. $(1, 5); (-6, -9)$.
6. $(8, \frac{1}{2}); (-1, -4)$.
7. $(1, -2); (-\frac{1}{2}, \frac{5}{2})$.
9. $(4 + 3i, 2 + i); (4 - 3i, 2 - i)$.
10. $(3 + \sqrt{7}, 1 - \sqrt{7}); (3 - \sqrt{7}, 1 + \sqrt{7})$.
11. $(2, \frac{3}{2}); (-4, 6)$.
13. $(1, 1); (\frac{1}{2}, \frac{1}{4})$.
14. $(0, 0); (10, -8)$.
15. $(\frac{1}{2}, 0); (-2, -1)$.
17. $(1 + 3\sqrt{2}, 1 - 5\sqrt{2}); (1 - 3\sqrt{2}, 1 + 5\sqrt{2})$.
18. $(\frac{3}{2}, \frac{1}{2}); (29, -16)$.
19. $(4, 1); (-4, -1); (i, -4i); (-i, 4i)$.

Exercise 45, Page 134

1. $(\frac{5}{2}, 4); (\frac{5}{2}, -4); (-\frac{5}{2}, 4); (-\frac{5}{2}, -4)$.
2. $(\sqrt{11}, \sqrt{2}); (\sqrt{11}, -\sqrt{2}); (-\sqrt{11}, \sqrt{2}); (-\sqrt{11}, -\sqrt{2})$.
3. $(3i, 2i); (3i, -2i); (-3i, 2i); (-3i, -2i)$.
5. $(\sqrt{11}, 5); (-\sqrt{11}, 5); (0, -6)$, double solution.
6. $(2, -3); (-2, -3); (3, 2); (-3, 2)$.
7. $(0, \sqrt{6}); (0, -\sqrt{6}); (2, \sqrt{5}); (2, -\sqrt{5})$.
9. $(\frac{1}{3}, \frac{14}{3}); (\frac{1}{2}, 3)$.
10. $(1, 1); (-\frac{1}{3}, -3)$.
11. $(5, -3); (-5, 3)$.
13. $(\sqrt{7}, 4); (-\sqrt{7}, 4); (i\sqrt{3}, -1); (-i\sqrt{3}, -1)$.
14. $(4, \sqrt{5}); (4, -\sqrt{5}); (5, \sqrt{11}); (5, -\sqrt{11})$.
15. $(5, 1); (-10, -\frac{8}{7})$.

Exercise 46, Pages 136–137

1. $(2, 2); (-2, -2); (2\sqrt{5}, -2\sqrt{5}); (-2\sqrt{5}, 2\sqrt{5})$.
2. $(6\sqrt{3}, 2\sqrt{3}); (-6\sqrt{3}, -2\sqrt{3}); (2i, -2i); (-2i, 2i)$.

3. $(4, 2)$; $(-4, -2)$; $(3\sqrt{2}, \sqrt{2})$; $(-3\sqrt{2}, -\sqrt{2})$.
5. $(1, 4)$; $(-1, -4)$; $(i, 2i)$; $(-i, -2i)$.
6. $(0, 2)$; $(0, -2)$; $(\frac{1}{3}, \frac{5}{3})$; $(-\frac{1}{3}, -\frac{5}{3})$.
7. $(2, 6)$; $(-2, -6)$; $(2, -4)$; $(-2, 4)$.
9. $(\sqrt{2}, \sqrt{2})$; $(-\sqrt{2}, -\sqrt{2})$; $(\frac{3}{2}, 2)$; $(-\frac{3}{2}, -2)$.
10. $(6, -3)$; $(-6, 3)$; $(5\sqrt{3}, 4\sqrt{3})$; $(-5\sqrt{3}, -4\sqrt{3})$.
11. $(1, 3)$; $(3, 1)$; $(i, -i)$; $(-i, i)$.
13. $(1, 3)$; $(3, 1)$; $(5, 7)$; $(7, 5)$.
14. $(0, 2)$; $(2, 0)$; $(3 + i, 3 - i)$; $(3 - i, 3 + i)$.
15. $(1 + \sqrt{2}, 1 - \sqrt{2})$; $(1 - \sqrt{2}, 1 + \sqrt{2})$; $(-4 + i, -4 - i)$; $(-4 - i, -4 + i)$.

Exercise 47, Pages 139–141

1. $(4, 3)$ tangency.
2. No real solution.
3. $(3, -2)$; $(-2, 3)$.
5. $(2.2, 3.5)$; $(-.9, -2.8)$.
6. $(6, 1)$; $(-2, -3)$.
7. $(2, 4)$; $(4, 2)$; $(-2, -4)$; $(-4, -2)$.
9. $(3.2, -4)$; $(-3.2, -4)$; $(0, 6)$ tangency.
10. $(3, 2.6)$; $(3, -2.6)$; $(-4, 0)$ tangency.
11. No real solution.
13. $(2, \sqrt{3})$; $(2, -\sqrt{3})$; $(1, \sqrt{5})$; $(1, -\sqrt{5})$.
14. $(0, 10)$; $(10, 0)$; $(-5 + i\sqrt{55}, -5 - i\sqrt{55})$; $(-5 - i\sqrt{55}, -5 + i\sqrt{55})$.
15. $(\frac{5}{2}, 0)$; $(-\frac{1}{2}, \frac{3}{2})$.
17. $(0, 1, 3)$; $(\frac{5}{3}, \frac{8}{3}, -\frac{1}{3})$.
18. $(1, 2, 3)$; $(1, 2, -3)$; $(3, 0, \sqrt{5})$; $(3, 0, -\sqrt{5})$.
19. $(3, -1)$; $(-1, 3)$.
21. 5 ft. by 41 ft.
22. 8 ft. by 9 ft.; 6 ft. square.
23. 16 ft. by 18 ft.; 12 ft. by 24 ft.
25. 14; 25; 25.
26. 4 hours; 6 hours.
27. 2 and 12.
29. $2\frac{2}{5}$ mph; $9\frac{3}{5}$ mph.
30. $2\frac{1}{7}$ mph; $32\frac{1}{7}$ mph.
31. 5 ft.; 212 ft.
33. 40 mph; 60 mph.
34. 3 hours.
35. 45 mph; 2 : 20 P.M.

Exercise 48, Pages 143–144

1. $\frac{3}{4}$. **2.** $\frac{21}{2}$. **3.** $\frac{32}{3}$. **5.** $\frac{1}{14}$.
6. $\frac{5}{24}$. **7.** $\frac{10}{3}$. **9.** 0; 7. **10.** $-\frac{1}{2}$.
11. ± 8. **13.** ± 6. **14.** $\pm a^5$. **15.** 15.
17. 23 ft. **18.** 36 ft. **19.** $1\frac{1}{4}$ in.; $1\frac{1}{2}$ in.; 2 in.
21. 27 sq. in. **22.** $73\frac{1}{3}$ miles.

Exercise 49, Pages 147–149

1. $A = \pi r^2$.
2. $A = 6E^2$.
3. $A = \frac{1}{2}h(b_1 + b_2)$.
5. $W = \dfrac{75x}{2d^2}$.
6. $U = \frac{8}{15}x\sqrt{t}$.
7. $R = \dfrac{16}{st^2}$.

9. (a) y is multiplied by 4; (b) y is multiplied by $\sqrt{2}$; (c) y is multiplied by $\frac{1}{8}$; (d) y is multiplied by $\dfrac{\sqrt{2}}{2}$.

10. (a) M is halved; (b) M is doubled; (c) M is multiplied by 4; (d) M remains the same.

11. 3.
13. $\frac{400}{3}$.
14. 128.
15. $\frac{56}{3}$.
17. 20 sq. ft.
18. 810 horsepower.
19. 54 horsepower.
21. $44\frac{4}{9}$ lb.
22. 240 ft.
23. 52 lb. per sq. in.
25. 3 days.
26. 105 horsepower.
27. 4 in.
29. 12 seconds.
30. $30\sqrt{30} = 164^+$ years.
31. A varies directly as C, directly as the square of r, and inversely as the square root of l.

Exercise 50, Pages 152–154

1. 7, 11.
2. $\frac{2}{3}, \frac{5}{6}$.
3. Not A.P.
5. $l = 206$; $S = 3135$.
6. $l = 1.65$; $S = 17.55$.
7. $l = 5.73$; $S = 58.65$.
9. $l = -15$; $S = -40$.
10. $l = 10\frac{1}{3}$; $S = 67\frac{5}{6}$.
11. $a = 19$; $S = 8619$.
13. $a = 3.4$; $d = -.1$.
14. $d = 8$; $l = 69$.
15. $d = -7$; $l = 2$.
17. $n = 13$; $l = 27$.
18. $n = 4$, $l = 1$; $n = 9$, $l = -\frac{2}{3}$.
19. $a = -5$; $l = 13$.
21. $n = 5$, $a = 1$; $n = 8$, $a = -\frac{1}{2}$.
22. $n = 11$; $a = -17$.
23. $n = 80$; $S = 46{,}360$.
25. $6\frac{1}{4}, 7\frac{1}{2}, 8\frac{3}{4}$.
26. $7\frac{1}{3}, 7\frac{2}{3}, 8, 8\frac{1}{3}, 8\frac{2}{3}$.
27. $4\frac{4}{5}, 6\frac{3}{5}, 8\frac{2}{5}, 10\frac{1}{5}$.
29. 1776.
30. $8173\frac{1}{2}$.
31. 69,700.
34. 3.
35. $S = \dfrac{n}{2}[2l - (n-1)d]$.
37. 433.
38. 190.
39. 600 ft.
41. \$600.
42. 44,550.

Exercise 51, Pages 156–157

1. $-500, 2500$.
2. Not G. P.
3. $6\sqrt{2}$, 12.
5. 24, 16.
6. $\frac{9}{16}, \frac{27}{32}$.
7. Not G. P.
9. $l = 192$; $S = 381$.
10. $l = 1$; $S = 2047$.

11. $l = -1701$; $S = -1274$.
13. $l = -6$; $S = 510$.
14. $l = -1024$; $S = -680$.
15. $l = 700{,}000$; $S = 777{,}777$.
17. $l = \frac{9}{16}$; $S = 1\frac{67}{144}$.
18. $l \doteq 3.1670$; $S = 75.40$.
19. $a = \frac{1}{2}$; $S = 129\frac{1}{2}$.
21. $a = 3$; $l = 48$.
22. $a = 2$; $l = 54$.
23. $n = 5$; $l = 162$.
25. $r = 3$; $n = 5$.
26. $r = -4$; $n = 4$.
27. $a = 64$; $n = 7$.
29. $5\sqrt[3]{2}$; $5\sqrt[3]{4}$.
30. $-48, 24, -12, 6$.
31. $\frac{1}{2}, 1, 2, 4, 8$; $-\frac{1}{2}, 1, -2, 4, -8$.
33. 20 or -20.
34. $\frac{8}{3}$ or $-\frac{8}{3}$.
35. $S = \dfrac{l - lr^n}{r^{n-1} - r^n}$.
37. 0; 5.
38. $r = 9, l = 81$; $r = -10, l = 100$.
39. \$25; \$20; \$16.
41. 131.22 gal.
42. 4 in.

Exercise 52, Pages 159–160

1. 9. **2.** 125. **3.** $\frac{3}{5}$. **5.** $\frac{9}{11}$.
6. $4 + 2\sqrt{2}$. **7.** $\frac{2}{9}$. **9.** $\frac{2}{11}$. **10.** $\frac{5}{11}$.
11. $\frac{113}{33}$. **13.** $\frac{68}{55}$. **14.** $\frac{312}{55}$. **15.** $\frac{55}{111}$.
17. $\frac{137}{111}$. **18.** $\frac{2}{7}$. **19.** 16. **21.** 140 ft.
22. 270.

Exercise 53, Pages 160–161

1. $\frac{2}{3}, 1$.
2. $\frac{1}{5}, -1$.
3. $3, \frac{12}{5}$.
5. $\frac{1}{83}$.
6. $\frac{12}{19}$.
7. $\frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \frac{1}{15}, \frac{1}{17}$.
9. $\frac{2}{5}, \frac{9}{19}, \frac{18}{31}$.
10. $\frac{4}{3}$.
11. $\dfrac{2ab}{a + b}$.
14. 10.

Exercise 54, Pages 161–163

1. 24. **2.** $\frac{1}{12}$. **3.** 27. **5.** 487; 12,100.
6. 480; 315. **7.** $\frac{1}{29}$. **9.** .81; 142.51. **10.** $\frac{1}{2}$.
11. 200. **13.** 8190. **14.** $\frac{243}{100{,}000}$. **15.** 15.
17. 1540 ft.
18. (*a*) \$2525; (*b*) \$25,350.
19. 5740.
21. (*a*) $33\frac{77}{125}$ in.; (*b*) 50 in.
22. $2\frac{1}{2}, 7\frac{1}{2}, 22\frac{1}{2}$; $20, -10, 5$.
23. 95.5.
25. \$6100.
26. \$3000, \$3400, \$3800, \$4200, \$4600, \$5000.
27. The second.
29. 1, 4, 7, 10, 13, 16; 1, 1, 1, 1, 1, 1.
30. 2 and 8.

Exercise 56, Pages 170–171

1. $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$.
2. $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$.
3. $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$.
5. $r^8 - 4r^6t^3 + 6r^4t^6 - 4r^2t^9 + t^{12}$.
6. $x^8 - 1.2x^5 + .54x^2 - \dfrac{.108}{x} + \dfrac{.0081}{x^4}$.
7. $32s^{15} + 80s^{12} + 80s^9 + 40s^6 + 10s^3 + 1$.
9. $64x^{21} + 240x^{14}y + 300x^7y^2 + 125y^3$.
10. $\dfrac{32x^5}{y^5} + \dfrac{40x^4}{y^3} + \dfrac{20x^3}{y} + 5x^2y + \dfrac{5xy^3}{8} + \dfrac{y^5}{32}$.
11. $\dfrac{x^4}{y^4} - \dfrac{2x^3}{y^2} + \dfrac{3x^2}{2} - \dfrac{xy^2}{2} + \dfrac{y^4}{16}$.
13. $x^{20} - 50x^{16}y + 1000x^{12}y^2 - 10{,}000x^8y^3 + 50{,}000x^4y^4 - 100{,}000y^5$.
14. $x^3 - 6x^{\frac{5}{2}}y^{\frac{1}{4}} + 15x^2y^{\frac{1}{2}} - 20x^{\frac{3}{2}}y^{\frac{3}{4}} + 15xy - 6x^{\frac{1}{2}}y^{\frac{5}{4}} + y^{\frac{3}{2}}$.
15. $576 + 256\sqrt{5}$.
17. $x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + \dfrac{56}{x^2} + \dfrac{28}{x^4} + \dfrac{8}{x^6} + \dfrac{1}{x^8}$.
18. $r^3 + s^3 + t^3 + 3r^2s + 3rs^2 + 3r^2t + 3rt^2 + 3s^2t + 3st^2 + 6rst$.
19. $a^{40} + 40a^{39}b + 780a^{38}b^2 + 9880a^{37}b^3 + \cdots$.
21. $x^{105} - 42x^{100}y + 840x^{95}y^2 - 10{,}640x^{90}y^3 + \cdots$.
22. $x^8 - \frac{8}{3}x^7 + \frac{28}{9}x^6 - \frac{56}{27}x^5 + \cdots$.
23. $r^{60} - 30r^{54}\sqrt{t} + 405r^{48}t - 3240r^{42}t\sqrt{t} + \cdots$.
25. $64 + 96s + 60s^2 + 20s^3 + \cdots$.
26. $\dfrac{x^{26}}{y^{26}} + \dfrac{13x^{23}}{y^{23}} + \dfrac{78x^{20}}{y^{20}} + \dfrac{286x^{17}}{y^{17}} + \cdots$.
27. $x^{48} + 120x^{44}y + 6600x^{40}y^2 + 220{,}000x^{36}y^3 + \cdots$.
29. $-c^{22} + 11c^{20}d^3 - 55c^{18}d^6 + 165c^{16}d^9 - \cdots$.
30. $-r^{-14} - 7r^{-12}b^{-1} - 21r^{-10}b^{-2} - 35r^{-8}b^{-3} - \cdots$.
31. 1.0212.
33. .8864.
34. .9608.
35. $\frac{1}{132}$.
37. 455.
38. 98.
39. $k + 1$.

Exercise 57, Page 174

1. $165a^8w^{15}$.
2. $2860a^9$.
3. $-\dfrac{252x^4}{y^5}$.
5. $-6435x^8y^7$.
6. $792a^7b^5$.
7. $56x^5y^{30}$.
9. $4320x^3y^3$.
10. $-35{,}000x^8$; $350{,}000x^6$.
11. $1 - x + x^2 - x^3 + \cdots$.
13. $1 + 2x + 3x^2 + 4x^3 + \cdots$.
14. $1 - \dfrac{x}{2} + \dfrac{3x^2}{8} - \dfrac{5x^3}{16} + \cdots$.
15. $x + \dfrac{y}{2x} - \dfrac{y^2}{8x^3} + \dfrac{y^3}{16x^5} - \cdots$.

17. $125 + \frac{15y}{2} + \frac{3y^2}{40} - \frac{y^3}{2000} + \cdots.$

18. $10 - \frac{x^2}{20} - \frac{x^4}{8000} - \frac{x^6}{1,600,000} - \cdots.$ 19. 1.0099.

21. 1.0098. 22. 1.0196. 23. .98985.

25. 2.05. 26. 2.005. 27. 2.01.

Exercise 58, Pages 179–180

1. $x < 2$. 2. $x > 9$. 3. $x > 50$.
5. $x < 6$. 6. $x < -4$. 7. $x > \frac{1}{3}$.
9. $x > \frac{2}{9}$. 10. $x > -12$. 11. $x > 3$ or $x < -3$.
13. $0 < x < 2$. 14. $x > 4$ or $x < -2$. 15. $1 < x < 4$.
17. $x > 4.4$ or $x < -.4$. 18. $1.6 < x < 4.4$.
19. $x > 1$ or $x < -\frac{1}{2}$. 21. $-1 < x < 1$ or $x > 3$.
22. $.7 < x < 2.9$ or $x < -.5$. 23. $x < 1.2$.
25. $-1 < x < 7$. 26. $x > 8$ or $x < -4$.
27. $x \geqq 6$ or $x \leqq -8$. 29. $x > 7$ or $x < -7$.
30. $-9 < x < 9$. 31. $-\frac{3}{2} \leqq x \leqq \frac{3}{2}$.
37. $k > 2$ or $k < 0$. 38. $k > \frac{25}{12}$.
39. $-6 < k < 6$.

Exercise 59, Pages 183–184

1. $11 - 4i$. 2. $6 + 4i$. 3. $(4 + \sqrt{5}) - 11i$.
5. $16i$. 6. $2 + 2i$. 7. $2 - i$.
9. $5 + 3i$. 10. -6. 11. -12.
13. $41 - 62i$. 14. $-11 + 2i$. 15. $-4 + 5i\sqrt{15}$.
17. $-11 - 60i$. 18. $4 - 6i\sqrt{5}$. 19. $-59 - 12i\sqrt{7}$.
21. -4. 22. $-85 + 20i$. 23. $-1 + i$.
25. $1 - 4i$. 26. $7 + 2i$. 27. $\frac{11}{50} - \frac{23}{50}i$.
29. $\frac{4}{11} + \frac{6i\sqrt{2}}{11}$. 30. $8 - 7i$. 31. $-\frac{4}{5} + \frac{3}{5}i$.
33. $-\frac{226}{13} + \frac{207}{13}i$. 34. $-\frac{51}{85} + \frac{68}{85}i$. 35. $\frac{4}{17} - \frac{1}{17}i$.
37. i. 38. $-\frac{i\sqrt{5}}{25}$. 39. $-5 + 7i$.
41. $-11i$. 42. $6 - 2i\sqrt{5}$. 43. $x = 3; y = 7$.
45. $x = 2; y = 7$.
46. $x = 5, y = 3; x = -\frac{3}{2}, y = -10$.
47. $x = 4; y = 9$.

Exercise 60, Pages 188–189

9. $5\sqrt{2}(\cos 315° + i \sin 315°)$.
10. $2\sqrt{2}(\cos 45° + i \sin 45°)$.
11. $2(\cos 135° + i \sin 135°)$.
13. $5(\cos 180° + i \sin 180°)$.
14. $7(\cos 270° + i \sin 270°)$.
15. $4(\cos 90° + i \sin 90°)$.
17. $2(\cos 60° + i \sin 60°)$.
18. $4(\cos 300° + i \sin 300°)$.
19. $2\sqrt{3}(\cos 210° + i \sin 210°)$.
21. $13(\cos 112.6° + i \sin 112.6°)$.
22. $5(\cos 233.1° + i \sin 233.1°)$.
23. $\sqrt{5}(\cos 296.6° + i \sin 296.6°)$.
25. $-7i$.
26. -5.
27. $-4 + 4i\sqrt{3}$.
29. $\frac{\sqrt{3}}{2} - \frac{3}{2}i$.
30. $2\sqrt{3} + 2i$.
31. $-5\sqrt{2} - 5i\sqrt{2}$.
33. $2.270 + 4.455i$.
34. $-.174 + .985i$.
35. (*a*) 0°; (*b*) 180°; (*c*) 90°; (*d*) 270°.

Exercise 61, Pages 191–192

1. $28i$.
2. 8.
3. -15.
5. $-5.74 + 8.19i$.
6. $4 + 4i\sqrt{3}$.
7. $\frac{3}{2} - \frac{3i\sqrt{3}}{2}$.
9. $-1 - i\sqrt{3}$.
10. $2i$.
11. $-2\sqrt{3} + 2i$.
17. $4 + 4i\sqrt{3}$.
18. -36.
19. $-125i$.
21. $4 + 4i$.
22. -324.
23. $-512i$.
25. $-72 - 72i\sqrt{3}$.
26. -1.
27. $-\frac{1}{2} - \frac{i\sqrt{3}}{2}$.
29. $-116.75 + 44.75i$, using three-place tables. The correct value is $-117 + 44i$.
30. $-527.5 + 335.0i$, using three-place tables. The correct value is $-527 + 336i$.
31. $28.2 + 95.9i$, using three-place tables. The correct value is $28 + 96i$.

Exercise 62, Pages 193–194

1. $2(\cos 50° + i \sin 50°) = 1.286 + 1.532i$;
$2(\cos 140° + i \sin 140°) = -1.532 + 1.286i$;
$2(\cos 230° + i \sin 230°) = -1.286 - 1.532i$;
$2(\cos 320° + i \sin 320°) = 1.532 - 1.286i$.
2. $2(\cos 61° + i \sin 61°) = .970 + 1.750i$;
$2(\cos 133° + i \sin 133°) = -1.364 + 1.462i$;
$2(\cos 205° + i \sin 205°) = -1.812 - .846i$;
$2(\cos 277° + i \sin 277°) = .244 - 1.984i$;
$2(\cos 349° + i \sin 349°) = 1.964 - .382i$.

3. $\cos 22° + i \sin 22° = .927 + .375i$;
$\cos 82° + i \sin 82° = .139 + .990i$;
$\cos 142° + i \sin 142° = -.788 + .616i$;
$\cos 202° + i \sin 202° = -.927 - .375i$;
$\cos 262° + i \sin 262° = -.139 - .990i$;
$\cos 322° + i \sin 322° = .788 - .616i$.

5. $-4\sqrt{2} + 4i\sqrt{2}; 4\sqrt{2} - 4i\sqrt{2}$. **6.** $1 + i; -1 - i$.

7. $-2\sqrt{3} + 2i; 2\sqrt{3} - 2i$.

9. $2(\cos 15° + i \sin 15°) = 1.932 + .518i$;
$2(\cos 135° + i \sin 135°) = -\sqrt{2} + i\sqrt{2}$;
$2(\cos 255° + i \sin 255°) = -.518 - 1.932i$.

10. $\cos 75° + i \sin 75° = .259 + .966i$;
$\cos 195° + i \sin 195° = -.966 - .259i$;
$\cos 315° + i \sin 315° = \frac{\sqrt{2}}{2} - \frac{i\sqrt{2}}{2}$.

11. $5 + 5i\sqrt{3}; -10; 5 - 5i\sqrt{3}$.

13. $4(\cos 75° + i \sin 75°) = 1.036 + 3.864i$;
$4(\cos 165° + i \sin 165°) = -3.864 + 1.036i$;
$4(\cos 255° + i \sin 255°) = -1.036 - 3.864i$;
$4(\cos 345° + i \sin 345°) = 3.864 - 1.036i$.

14. $2\sqrt{2} + 2i\sqrt{2}; -2\sqrt{2} + 2i\sqrt{2}; -2\sqrt{2} - 2i\sqrt{2}; 2\sqrt{2} - 2i\sqrt{2}$.

15. $1 + i\sqrt{3}; -\sqrt{3} + i; -1 - i\sqrt{3}; \sqrt{3} - i$.

17. $5(\cos 53.1° + i \sin 53.1°) = 3.005 + 4.000i$ and
$5(\cos 233.1° + i \sin 233.1°) = -3.005 - 4.000i$,
using three-place tables. The correct values are $3 + 4i$ and $-3 - 4i$.

18. $2(\cos 22.7° + i \sin 22.7°) = 1.846 + .772i$;
$2(\cos 142.7° + i \sin 142.7°) = -1.592 + 1.212i$;
$2(\cos 262.7° + i \sin 262.7°) = -.254 - 1.982i$.

19. $\cos 0° + i \sin 0° = 1$;
$\cos 72° + i \sin 72° = .309 + .951i$;
$\cos 144° + i \sin 144° = -.809 + .588i$;
$\cos 216° + i \sin 216° = -.809 - .588i$;
$\cos 288° + i \sin 288° = .309 - .951i$.

21. $\sqrt{3} + i; 2i; -\sqrt{3} + i; -\sqrt{3} - i; -2i; \sqrt{3} - i$.

22. $10(\cos 18.4° + i \sin 18.4°) = 9.49 + 3.16i$;
$10(\cos 198.4° + i \sin 198.4°) = -9.49 - 3.16i$.

Exercise 63, Pages 199–200

1. $3x^3 + 4x^2 - 2x - 10; R = -41$. **2.** $5x^3 - 10x^2 + 3x - 6; R = 9$.
3. $x^2 - 6x - 6; R = 3$. **5.** $2x^2 - 3x + 5; R = 0$.
6. $3x^2 - 2x; R = 7$. **7.** $3x^3 + 2x^2 + 4x - 13; R = 12$.

9. $8x^2 - 6; R = -10.$
10. $6x^3 - 3x^2 + 9; R = -11.$
11. $x^2 - .8x + 1.76; R = -1.182.$
13. $f(2) = -5; f(-3) = 80.$
14. $f(4) = 392; f(-1) = -8.$
15. $g(1) = -4; g(-2) = -37.$

Exercise 64, Pages 202–203

9. $-1; 2; 4.$
10. $-1; 1; 4.$
11. $-2.9; .5; 1.5.$
13. $.4; 2.6.$
14. $-2; .6; 2; 3.4.$
15. $-2; 0; 1; 3.$

Exercise 65, Pages 207–208

1. $-5; -5; 1; 3 \pm \sqrt{7}.$
2. $-4; \frac{3}{7}; \frac{3}{7}; \pm 5i.$
3. $-8; \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; -5 \pm 2i.$
5. $0; 0; \pm\sqrt{7}; \pm\frac{2}{3}i.$
6. $-9; -9; -9; 6; 6; \dfrac{3 \pm \sqrt{7}}{2}.$
7. $x^4 - 5x^3 + 3x^2 + 9x = 0.$
9. $8x^3 - 10x^2 - x + 3 = 0.$
10. $21x^5 + 20x^4 - 3x^3 - 2x^2 = 0.$
11. $x^3 + 2x^2 - 11x - 22 = 0.$
13. $x^3 - 3x^2 + x - 3 = 0.$
14. $x^3 + 9x^2 + 4x + 36 = 0.$
15. $4x^3 - 8x^2 + 41x - 37 = 0.$
17. $x^5 + 3x^4 - 30x^3 - 14x^2 = 0.$
18. $x^3 - 4x^2 + 4x - 1 = 0.$
19. $x^5 - 2x^4 + x^3 = 0.$
21. $x^3 - 4x^2 - 10x - 12 = 0.$
22. $x^4 - 6x^3 + 11x^2 - 6x + 10 = 0.$
25. $3; 3; 1 \pm \sqrt{6}.$
26. $-1; -1; 3 \pm i.$

Exercise 67, Page 212

1. 2 pos., 1 neg.; or 1 neg., 2 imag.
2. 1 neg., 2 imag.
3. 1 pos., 2 neg.; or 1 pos., 2 imag.
5. 1 pos., 3 neg.; or 1 pos., 1 neg., 2 imag.
6. 2 pos., 2 neg.; or 2 pos., 2 imag.; or 2 neg., 2 imag.; or 4 imag.
7. 1 pos., 1 neg., 2 imag.
9. 6 imag.
10. 3 pos., 1 neg.; or 1 pos., 1 neg., 2 imag.
11. 1 pos., 2 neg., 2 imag.; or 1 pos., 4 imag.
13. 2 pos., 1 neg., 4 imag.; or 1 neg., 6 imag.
14. 3 pos., 1 neg., 2 imag.; or 1 pos., 1 neg., 4 imag.
15. 1 pos., 1 neg., and 2 zero roots.

Exercise 68, Page 213

1. $u = 2; l = -1.$
2. $u = 1; l = -1.$
3. $u = 5; l = -1.$
5. $u = 2; l = 0.$
6. $u = 0; l = -2.$
7. $u = 3; l = -1.$
9. $u = 2; l = -3.$
10. $u = 3; l = -1.$
11. $u = 1; l = -2.$

Exercise 69, Page 217

1. $2; 5 \pm \sqrt{21}$.
2. $-2; -2; -3$.
3. $3; -1; -2$.
5. $\frac{1}{3}; -\frac{1}{5}; -1$.
6. $\frac{1}{5}; \frac{4 \pm 3\sqrt{2}}{2}$.
7. $-\frac{1}{6}; \frac{-4 \pm \sqrt{13}}{3}$.
9. No rational root.
10. No rational root.
11. $\frac{2}{3}; 1 \pm i$.
13. $0; 2; 3; -5$.
14. $2; 6; -4$.
15. $\frac{1}{2}; -\frac{1}{4}; -3$.
17. $-\frac{7}{2}; -1 \pm \sqrt{3}$.
18. $\frac{5}{9}; 2 \pm \sqrt{5}$.
19. $-1; -1; -1; -1$.
21. $\frac{1}{3}; \frac{1}{3}; 1 \pm i$.
22. $-\frac{1}{3}; -\frac{1}{2}; \pm i\sqrt{5}$.
23. $-4; -2 \pm \sqrt{10}$.
25. $\frac{1}{7}; -1; \pm \sqrt{3}$.
26. $-\frac{1}{3}; -3; -4 \pm \sqrt{19}$.
27. $\frac{1}{5}; 1; 1; \pm i\sqrt{2}$.
29. $1; -\frac{1}{2}; \frac{-1 \pm \sqrt{33}}{4}$.
30. $\frac{1}{2}; \frac{1}{2}; \frac{3}{2}; -2$.

Exercise 70, Page 220

1. .14.
2. .11.
3. 2.06.
5. .38; 2.62.
6. -2.94; .46; 1.47.
7. .198.
9. .229.
10. 1.255.
11. 1.306.
13. .32; .76.
14. -2.1; .3; 1; 1.9.
15. .7; 2.

Exercise 71, Page 222

1. $y^3 - 3y^2 - 4y + 12 = 0$.
2. $y^3 - 7y^2 - 6y + 72 = 0$.
3. $5y^3 + 9y^2 + 3y - 6.72 = 0$.
5. $y^4 - y^2 + 8y + 1 = 0$.
6. $y^4 - 12y^3 + 47y^2 - 58y - 5 = 0$.
7. $2y^4 + 23y^3 + 84y^2 + 88y - 25 = 0$.
9. (*a*) $y^3 + 13y^2 + 53y - 15 = 0$;
(*b*) $y^3 + 13.6y^2 + 58.32y - 3.872 = 0$;
(*c*) $y^3 + 13.78y^2 + 59.9628y - .323624 = 0$.

Exercise 72, Pages 225–226

1. 2.453.
2. 2.135.
3. 1.612.
5. 1.061.
6. 4.404.
7. .065; 3.463; 4.473.
9. .807; 6.193.
10. 2.268; 5.732.
11. -1.541; 4.541.
13. .219; 1.586; 2.281; 4.414.
14. .229; 1.
15. .198; 2.
17. 1.710.
18. 2.748.
19. .574 in.

Exercise 73, Page 227

1. $x^4 - (r_1 + r_2 + r_3 + r_4)x^3 + (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4)x^2 - (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)x + r_1r_2r_3r_4 = 0.$
2. $2x^4 - 19x^3 + 61x^2 - 74x + 24 = 0.$
3. $x^4 - 7x^2 + 6x = 0.$
5. $3x^3 - 23x^2 + 28x + 12 = 0.$
6. $x^5 - 12x^4 + 52x^3 - 102x^2 + 91x - 30 = 0.$
7. (a) $-\frac{5}{4}$; (b) $-\frac{3}{2}$; (c) $\frac{7}{4}$.
9. $2; p = 31; q = -30.$
10. $\frac{3}{5}; p = -33; q = 58.$
11. $2, -2, \frac{7}{2}; q = 28.$
13. $3, -6, 12; k = -9.$
14. $3, 3, 5, 5; p = -240; q = 225.$

Exercise 74, Pages 229–230

1. 2. 2. 3. 3. 1. 5. -1.
6. -2. 7. 0. 9. -2. 10. $\frac{1}{2}$.
11. $\frac{1}{3}$. 13. $\frac{3}{2}$. 14. 4. 15. $-\frac{1}{3}$.
17. 128. 18. $\frac{1}{125}$. 19. $\frac{1}{9}$. 21. 4.
22. 4. 23. 16. 25. $\frac{1}{81}$. 26. 16.
27. $\frac{1}{4}$. 29. $\log_3 243 = 5$. 34. $a^z = w$.
37. True. 38. True. 39. True.

Exercise 75, Pages 231–232

1. 0.78. 2. 1.33. 3. -0.37. 5. 1.80.
6. 2.40. 7. 0.06. 9. 0.19. 10. -0.17.
11. 3.85. 13. 0.815. 14. -1.70. 15. 0.70.
17. $\log (x^2 - 1)$. 18. $\log (6x - 4)$. 19. $\log \dfrac{a+b}{b}$. 21. $\log \dfrac{x^7y^5}{z^3}$.
22. $\log \dfrac{a^2b}{cd^5}$. 23. $\log 2\pi \sqrt{\dfrac{l}{g}}$. 25. True. 26. False.
27. False. 29. False. 30. True. 31. True.
33. False. 34. False.

Exercise 76, Page 236

1. 2.8228. 2. $9.9400 - 10$. 3. $8.9400 - 10$.
5. $6.9400 - 10$. 6. 1.8228. 7. 3.8228.
9. $9.8228 - 10$. 10. 5.9400. 11. 6.9400.
13. 26.3. 14. 8300. 15. 0.0830.
17. 0.00263. 18. 0.000263. 19. 26300.
21. 0.830. 22. 8.30. 23. 8,300,000.
25. 0.263. 26. 0.830. 27. 0.00830.

Exercise 77, Page 237

1. 1.9425. **2.** 3.5378. **3.** 9.8525 − 10. **5.** 7.3201 − 10.
6. 6.9196 − 10. **7.** 5.6503. **9.** 0.0170. **10.** 9.7924 − 10.
11. 332. **13.** 0.0543. **14.** 0.246. **15.** 0.000893.
17. 1.11. **18.** 6250 **19.** 0.278. **21.** 0.636.
22. 0.0289.

Exercise 78, Pages 239–240

1. 0.7354. **2.** 1.5076. **3.** 8.6183 − 10. **5.** 6.9496 − 10.
6. 7.9348 − 10. **7.** 2.4340. **9.** 9.1679 − 10. **10.** 8.3510 − 10.
11. 5.0902. **13.** 2.8453. **14.** 0.6951. **15.** 7.9763 − 10.
17. 30.27. **18.** 9.425. **19.** 0.2413. **21.** 0.004511.
22. 0.0001503. **23.** 554.8. **25.** 12,340. **26.** 603,400.
27. 0.08596. **29.** 0.00001822. **30.** 0.001078. **31.** 1.540.

Exercise 79, Pages 244–246

1. 167. **2.** 12.5. **3.** 474.
5. 1340. **6.** 178. **7.** 0.0908.
9. 0.0794. **10.** 3,480,000,000. **11.** 3.21.
13. 7,640,000. **14.** 0.000193. **15.** 9.86.
17. 0.0397. **18.** 0.0120. **19.** 0.678.
21. −0.000,000,000,00509. **22.** −1.98.
23. −7.07. **25.** 6.19. **26.** 408,000. **27.** 11.0.
29. 3.97. **30.** 4.93. **31.** 162. **33.** 1609.
34. 4027. **35.** 0.1558. **37.** 13.62. **38.** 2.517.
39. 0.3206. **41.** 8.774. **42.** 0.8674. **43.** 3.339.
45. 0.000438. **46.** 0.305. **47.** 0.33. **49.** 0.0000105.
50. 161. **51.** 0.387. **53.** 0.410. **54.** 7.36.
55. 1.325. **57.** 2803. **58.** 0.5385. **59.** 1.53 sec.

Exercise 80, Page 247

1. 1.61. **2.** 0.730. **3.** −1.31. **5.** −0.271.
6. −0.804. **7.** 4.10. **9.** 4.60. **10.** 1.02.
11. −2.974. **13.** 1.20. **14.** 3.33. **15.** 3.12.
17. 6.32; −0.32. **18.** $x = 1.76$; $y = 3.61$.
21. $y = 700x^3$. **22.** $y = \dfrac{10^{x+8}}{3}$.
23. $y = 9e^{-5x}$.

Exercise 81, Page 248

All results are written with four-figure accuracy.

1. 4.145.
2. 6.447.
3. -2.349.
5. -2.088.
6. -2.003.
7. 6.021.
9. 0.3892.
10. 1.042.
11. -3.001.

Exercise 82, Pages 251–252

1. \$24; \$1224.
2. \$36; \$836.
3. \$11.20; \$651.20.
5. 5 months.
6. 8%.
7. \$591; \$191.
9. \$1741; \$741.
10. \$4034.70; \$1034.70.
11. \$277.63.
13. \$5512.60.
14. \$1974.52.
15. 35 years.
17. \$6775.70.
18. 8.24%.
19. 3.5%.

Exercise 83, Pages 255–257

Note. All answers were obtained by the use of tables in this book. All significant figures after the first four are not reliable and may vary according to the method of solution.

1. \$9048.16; \$2773.78.
2. \$7289.22; \$4905.42.
3. \$3967.68; \$1533.66.
5. \$525.63.
6. \$4031.38.
7. \$1490.76.
9. \$19,600.40.
10. The second; first, \$811.09; second, \$840.25.
11. \$223.35.
13. \$4709.76.
14. \$9290.24.
15. \$1554.06.
17. \$664.75.
18. \$28,501$^+$.
19. \$239.02.

Exercise 84, Pages 259–260

1. 132.
2. 30.
3. 36.
5. 240.
6. 6,759,324.
7. (*a*) 12,144; (*b*) 13,824.
9. 72.
10. 24.
11. 15,120.
13. 8.
14. 243.
15. 48.

Exercise 85, Pages 262–263

1. 1320.
2. 9900.
3. 20.
5. 40,320.
6. 9240.
7. 360.
9. 2880.
10. (*a*) 10,080; (*b*) 4320.
11. 1440.
13. 144.

14. 240.
15. 12.
17. (*a*) 210; (*b*) 3360.
18. 10.
19. 1260.

Exercise 86, Pages 265–266

1. (*a*) 161,700; (*b*) 161,700.
2. 10.
3. 66.
5. 190.
7. 252.
9. 1260.
10. 37,800.
11. 3420.
13. 15.
14. 112.
15. 20.
17. 2,598,960.
18. 635,013,559,600.
19. 290,004.

Exercise 87, Pages 266–267

1. 456,976.
2. 1540.
3. 616.
5. 252.
6. 120.
7. 90.
9. 210.
11. 420.
13. 4320.
14. 25,350.
15. 9295.
17. 1680.
18. 72.
19. 288.

Exercise 88, Pages 271–272

1. $\frac{5}{13}$.
2. $\frac{31}{365}$.
3. \$1.50.
5. (*a*) $\frac{5}{22}$; (*b*) $\frac{4}{33}$; (*c*) $\frac{15}{22}$.
6. (*a*) $\frac{7}{24}$; (*b*) $\frac{21}{40}$; (*c*) $\frac{119}{120}$.
7. (*a*) $\frac{1}{17}$; (*b*) $\frac{13}{102}$; (*c*) $\frac{32}{663}$.
9. $\frac{9}{47}$
10. $\frac{8}{47}$.
11. 25.
13. $\dfrac{69,804}{93,362}$.
14. $\dfrac{2,391}{38,569}$.
15. $\dfrac{3,605}{92,637}$.
17. 74th year.
18. $\frac{1}{126}$.
19. (*a*) $\frac{5}{18}$; (*b*) $\frac{5}{6}$.
21. $\frac{5}{36}$.
22. $\frac{1}{3}$.

Exercise 89, Pages 275–276

1. $\frac{3}{4}$.
2. $\frac{1}{18}$.
3. $\frac{1}{8}$.
5. (*a*) $\frac{3}{8}$; (*b*) $\frac{9}{40}$; (*c*) $\frac{19}{40}$.
6. (*a*) $\frac{1}{16}$; (*b*) $\frac{1}{52}$; (*c*) $\frac{1}{26}$.
7. $\frac{2}{9}$.
9. $\frac{2}{25}$.
10. $\frac{197}{3980}$.
11. $\frac{97}{100}$.

Exercise 90, Page 278

1. (*a*) $\frac{16}{81}$; (*b*) $\frac{32}{81}$; (*c*) $\frac{24}{81}$; (*d*) $\frac{8}{81}$; (*e*) $\frac{1}{81}$.
2. (*a*) .0756; (*b*) .3483.
3. $\frac{81}{125}$.
5. (*a*) $\frac{135}{512}$; (*b*) $\frac{47}{128}$.
6. (*a*) $\frac{15}{64}$; (*b*) $\frac{11}{32}$.
7. (*a*) $\frac{4}{5}$; (*b*) $\frac{1}{25}$; (*c*) $\frac{16}{125}$; (*d*) $\frac{48}{125}$.

Exercise 91, Pages 278–280

1. $\frac{1}{1461}$.
2. 5 to 3.
3. (a) $\frac{4}{663}$; (b) $\frac{8}{663}$; (c) $\frac{33}{221}$.
5. $\frac{1}{8}$.
6. (a) $\frac{1}{4}$; (b) $\frac{1}{8}$; (c) 0.
7. $\frac{2}{5}$.
9. 3 cents.
10. \$1.80.
11. $\frac{256}{625}$.
13. (a) $\frac{1}{10}$; (b) $\frac{17}{30}$.
14. $\frac{41}{60}$.
15. $\frac{1}{165}$.
17. $\frac{39}{80}$.
18. (b).
19. (a) $\dfrac{2{,}197}{499{,}800}$; (b) $\dfrac{2{,}197}{20{,}825}$.

Exercise 92, Pages 289–291

3. (a) $4(-)\begin{vmatrix} 6 & 8 \\ -1 & 2 \end{vmatrix} + 7(+)\begin{vmatrix} 3 & 5 \\ -1 & 2 \end{vmatrix} + 0(-)\begin{vmatrix} 3 & 5 \\ 6 & 8 \end{vmatrix}$; (b) -3.
5. -10.
6. 51.
7. -4.
9. 12.
10. 38.
11. 51.
13. -342.
14. 33.
15. 374.
17. $abc + abd + acd + bcd + abcd$.
19. $a_1b_2c_3d_4 + a_1b_3c_4d_2 + a_1b_4c_2d_3 + a_2b_1c_4d_3 + a_2b_3c_1d_4 + a_2b_4c_3d_1$
$+a_3b_1c_2d_4 + a_3b_2c_4d_1 + a_3b_4c_1d_2 + a_4b_1c_3d_2 + a_4b_2c_1d_3 + a_4b_3c_2d_1$
$-a_1b_2c_4d_3 - a_1b_3c_2d_4 - a_1b_4c_3d_2 - a_2b_1c_3d_4 - a_2b_3c_4d_1 - a_2b_4c_1d_3$
$-a_3b_1c_4d_2 - a_3b_2c_1d_4 - a_3b_4c_2d_1 - a_4b_1c_2d_3 - a_4b_2c_3d_1 - a_4b_3c_1d_2$.

Exercise 93, Pages 293–295

1. $x = \frac{1}{6}; y = \frac{1}{3}; z = \frac{1}{2}$.
2. $x = \frac{1}{2}; y = \frac{3}{2}; z = \frac{5}{2}$.
3. $x = -5; y = 10; z = 5; w = -5$.
5. $x = 3; y = 2; z = 1; w = 0$.
6. $x = 1; y = 2; z = 3; w = 4$.
7. $x = \frac{1}{2}; y = 0; z = -\frac{1}{2}; w = 1$.
9. $v = 1; w = 2; x = 3; y = 2; z = -1$.
10. $x = 3; y = 2; z = 1; v = 0; w = -1$.
11. $z = 6\frac{1}{8}$.

Exercise 94, Page 298

1. $x = 3a + 2,\ y = a - 1,\ z = a$, where a is arbitrary.
2. Inconsistent.
3. Inconsistent.
5. Inconsistent.
6. $x = 2a + 3,\ y = a + 2,\ z = a$, where a is arbitrary.
7. $x = 0,\ y = 1,\ z = 2$.
9. $x = a + 2,\ y = a + 1,\ z = 2a,\ w = a$, where a is arbitrary.
10. Only the trivial solution $x = y = z = 0$.
11. $x : y : z : w = 1 : 2 : 3 : 4$.
13. 8.
14. 9.

Exercise 95, Page 304

1. $\dfrac{7}{x-4}+\dfrac{2}{x+1}$.
2. $\dfrac{8}{x-5}-\dfrac{9}{2x-1}$.
3. $\dfrac{2}{5(3x-4)}-\dfrac{1}{5(4x+3)}$.
5. $\dfrac{5}{3x-2}+\dfrac{1}{x-4}+\dfrac{2}{x+1}$.
6. $-\dfrac{5}{2x-1}+\dfrac{4}{x-3}+\dfrac{2}{x+1}$.
7. $5x+3+\dfrac{6}{x+2}+\dfrac{1}{x-1}$.
9. $\dfrac{1}{x+2}+\dfrac{2}{x-5}+\dfrac{3}{(x-5)^2}$.
10. $\dfrac{3}{x-1}-\dfrac{2}{x-2}+\dfrac{7}{(x-2)^2}$.
11. $-\dfrac{3}{4x-1}+\dfrac{2}{x-1}+\dfrac{1}{(x-1)^2}$.
13. $\dfrac{7}{x-2}-\dfrac{9}{(x-2)^3}$.
14. $\dfrac{7}{x-2}-\dfrac{6}{x-1}+\dfrac{5}{(x-1)^3}$.
15. $\dfrac{17}{(x-7)^3}$.
17. $\dfrac{5}{x-1}+\dfrac{6}{(x-1)^2}-\dfrac{7}{(x-1)^3}-\dfrac{8}{(x-1)^4}$.
18. $-\dfrac{1}{x-4}+\dfrac{3}{(x-4)^2}+\dfrac{1}{x}+\dfrac{1}{x^2}+\dfrac{2}{x^3}$.
19. $\dfrac{1}{x+5}+\dfrac{2}{x-1}+\dfrac{3}{x-4}-\dfrac{12}{2x-3}$.

Exercise 96, Page 306

1. $\dfrac{4}{x-1}+\dfrac{5x+6}{x^2+3}$.
2. $\dfrac{2}{x-2}+\dfrac{-2x+1}{x^2+x+1}$.
3. $\dfrac{2}{x+5}+\dfrac{x-3}{2x^2-x+6}$.
5. $\dfrac{1}{x-1}-\dfrac{1}{x-2}+\dfrac{x}{3x^2+4}$.
6. $\dfrac{2}{x-3}+\dfrac{5}{(x-3)^2}-\dfrac{x+5}{x^2+4}$.
7. $\dfrac{1}{x-2}+\dfrac{-x+2}{x^2+2x+4}$.
9. $\dfrac{1}{x}-\dfrac{1}{x^2}-\dfrac{1}{x^3}+\dfrac{2x+5}{5x^2+1}$.
10. $\dfrac{3x+1}{x^2+1}+\dfrac{4x-1}{x^2+2}$.
11. $\dfrac{x}{x^2+3x+5}+\dfrac{-x+3}{x^2+7}$.
13. $\dfrac{5x+6}{x^2+3}+\dfrac{7x+10}{(x^2+3)^2}$.
14. $\dfrac{x}{4x^2+7}+\dfrac{-7x+5}{(4x^2+7)^2}$.
15. $\dfrac{1}{x^2+x+4}-\dfrac{x+4}{(x^2+x+4)^2}$.
17. $\dfrac{1}{x-2}+\dfrac{2x}{x^2+x+2}+\dfrac{-4x+2}{(x^2+x+2)^2}$.
18. $\dfrac{6}{x}-\dfrac{1}{x^2}-\dfrac{5x+20}{x^2+3x+1}+\dfrac{5x+4}{(x^2+3x+1)^2}$.
19. $\dfrac{3}{x^2+1}+\dfrac{4x+5}{(x^2+1)^3}$.

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